DOI: 10.51790/2712-9942-2022-3-2-2

ФОРМА ЗЕМЛИ: ГЕОГРАФИЯ, АСТРОНОМИЯ И ГЕОМЕТРИЯ

А. Пападопулос
Университет Страсбурга, Национальный центр научных исследований Франции, Страсбург, Франция, papadop@math.unistra.fr

Посвящается моему уважаемому другу В. Галкину

Аннотация: рассматривается, каким образом вопрос об установлении формы Земли привел к исследованиям в области дифференциальной геометрии сфероидов. Наша цель – показать, как вопросы географии приводили к появлению глубоких математических теорий.

Ключевые слова: геометрия сфероида, форма Земли, геодезия на выпуклых поверхностях, гидростатика.

Для цитирования: Пападопулос А. Форма Земли: география, астрономия и геометрия. *Успехи кибернетики*. 2022;3(2):11–23. DOI: 10.51790/2712-9942-2022-3-2-2.

THE FIGURE OF THE EARTH: GEOGRAPHY, ASTRONOMY AND GEOMETRY

A. Papadopoulos
University of Strasbourg, The French National Centre for Scientific Research, Strasbourg, France, papadop@math.unistra.fr

Dedicated to Valerii Galkin, with friendship and esteem

Abstract: we explain how the question of the figure of the Earth led to the study of the differential geometry of the spheroid. More generally, our aim is to illustrate the fact that deep mathematical theories were motivated by geographical questions.

Keywords: geometry of the spheroid, figure of the Earth, geodesics on convex surfaces, hydrostatics.

Cite this article: Papadopoulos A. The Figure of the Earth: Geography, Astronomy and Geometry. *Russian Journal of Cybernetics*. 2022;3(2):11–23. DOI: 10.51790/2712-9942-2022-3-2-2.

Introduction

In the last third of the seventeenth century, the belief that the Earth has a spherical shape, which was prevailing in the Western world since Greek Antiquity, started to be challenged, principally in France and in England. Two antagonistic theories arose at about the same time, concerning the figure of the Earth. One theory claimed that this figure is a spheroid (that is, an ellipsoid of revolution) which is oblate, that is, flattened at the Poles and elongated at the Equator, and the other one said that the Earth is a prolate spheroid, that is, elongated at the Poles and flattened at the Equator.

From the purely mathematical point of view, the belief that the Earth is a spheroid motivated to a large extent the development of the differential geometry of such a surface; this is the main point I would like to emphasize in the present paper. Before that, I start by some markers on the history of the problem of the figure of the Earth.

The plan of this paper is the following.

It begins with an overview of the problem of the figure of the Earth, as it was addressed starting in the last third of the 17th century, where two conflicting theories arose, the one saying that the Earth is an oblate spheroid and the other that it is prolate.

Then the reader is presented an overview of a paper by Leonhard Euler, addressed to the general public, in which he explains in simple words the physical ideas that were behind the various theories concerning the shape of the Earth.

We also survey some other works of Euler that are related to the figure of the Earth: First, two purely mathematical memoirs on the geometry and the trigonometry of the spheorid, then, a memoir on the impact of the spheroidal shape of the Earth on astronomical observations, and finally, a memoir on the gravitational force exerted by a planet which has the form of a spheroid.

The subsequent sections are concerned with the works of three prominent mathematicians on problems related to the spheroid, namely, Clairaut, Lagrange and Legendre. Each of them studied these questions from his own particular point of view: Clairaut as a specialist of hydrodynamics, Lagrange using analysis and the calculus of variations, and Legendre, introducing approximations techniques in geodesy.

We will also talk about Gauss whose work on the differential geometry of surfaces was motivated by his official position of geographer,

After that, we review some work of Jacobi who studied the spheroid in relation with his work on Abelian integrals.

In the end, we mention briefly the works of several other mathematicians who worked on the geometry of the spheroid, motivated by the question of the shape of the Earth.

The question of the figure of the Earth

Our chronicle starts in the year 1672, when the French astronomer Jean Richer was sent to Cayenne (a region in French Guiana, near the Equator) to observe the planet Mars and compute its parallax, with the aim of obtaining an estimate of the distance from this planet to the Earth. In doing his experiments, Richer noticed that the length of a pendulum performing one beat per second is longer in Paris than near the Equator. He concluded that the value of the gravitational force is smaller in Cayenne than in Paris. This showed that the factor representing the acceleration of gravity which appears in the formula for the period of the pendulum is not uniform all around the Earth but depends on the latitude. The theory saying that the Earth is spherical[1] started being questioned.

A few years later, Christiaan Huygens, who was certainly the most celebrated seventeenth-century mathematician living in Paris, emitted the theory that the particles constituting the Earth, in addition to the fact that they are subject to the gravitational attraction force, are submitted to a centrifugal force due to the rotation of the Earth around its axis, and he concluded that in order to reach an equilibrium under the action of these two forces, the Earth has necessarily the form of an oblate spheroid.

At about the same time, Isaac Newton, in the 1687 edition of his *Principia* (Book III, Propositions XVIII–XX), based on his theory of universal gravitation, declared that the Earth has the form of an oblate spheroid, confirming Huygen's conclusion (but not necessarily his theories). Newton's claim was based on his assumption that the Earth was originally a fluid having a spherical shape, and that it acquired gradually a spheroidal shape under the effect of the mutual attraction force exerted between its various parts combined with the effect of the Earth's rotation around its axis. In fact, Newton provided a precise estimate for the flattening of the Earth, namely, he claimed that if a and b denote respectively the major a and the minor axis of this spheroid, then $\frac{a-b}{a} = \frac{1}{230}$. This estimate is different from the one that Huygens obtained. The latter, who did experiments using water pillars for the determination of the centrifugal force, and under the assumption that gravity is constant at the interior of the Earth, concluded that the Earth is more flattened at the Poles than what Newton thought.

Soon after, in France, the geographers and astronomers, led by Jean-Dominique Cassini[2] and based on a series of measurements that were conducted in the context of a project of land surveying, concluded that on the contrary, the Earth has the form of a prolate spheroid, that is, a spheroid elongated at the Poles and flattened at the Equator.

Each of the two points of view was defended vehemently, in England by the scientific community that was led by Newton, and in France by the astronomers of the Cassini family and other members of the Royal Academy of Sciences.

The dispute between the supporters of the two theories lasted several years, until the Royal Academy, together with the ministries of Marine and of Finances, and under the pressure of Pierre Louis de Maupertuis[3],

[1] In all our discussion here, we are neglecting the differences in altitude on the surface of the Earth due to the existence of mountains, valleys, etc., these differences being very small compared to the diameter of the Earth.

[2] Jean-Dominique Cassini (1625–1712) was a leading astronomer of Italian origin, who settled in Paris. He was a member of the Royal Academy of Science and the director of the Paris Observatory. He was the first of an unusually long list of leading geographers and astronomers from the same family.

[3] Pierre Louis de Maupertuis (1698–1759) was one of the main supporters in France of the theory saying that the Earth is oblate. More generally, he was a supporter of Newton's ideas on physics and philosophy, against the majority of the French academicians. We mention incidentally that in those times, there was a competition in France between Newton's and Descartes' ideas on subjects such as matter and attraction, and the dispute regarding the figure of the Earth was part of the debate between the supporters of the two theories, a debate that sometimes took the form of a conflict.

the famous French philosopher, mathematician, astronomer and naturalist, who was a distinguished member of this Academy, organized an expedition to Swedish Lapland whose objective was to measure the length of a degree of meridian in these regions close to the North Pole, in order to settle this question. Indeed, by the comparison between the result of these measurements with the length of a degree of a meridian at some known place far from the Pole (like Paris) would give information on whether the Earth is flattened or elongated at the North Pole.

The expedition took place in 1736–1737, under the leadership of Maupertuis, and including major scientists like the mathematician Alexis-Claude Clairaut and the Swedish mathematician, astronomer, geographer and meteorologist Anders Celsius. The results of this expedition were probably the most conclusive factors at that time regarding the question of the figure of the Earth.

Another expedition headed to Peru, the year before the one to Lapland, to make similar measures near the Equator. The Peru expedition lasted 9 years (1735 to 1744).

Euler's report on the figure of the Earth

I will start by an overview of an article by Euler on the question of the figure of the Earth, written for the general public. As a matter of fact, this question was not only discussed in scientific milieux, but also in the general cultured society of Saint Petersburg, Paris and other places, and it was not unnatural that Euler, who was a prolific writer, interested in all aspects of theoretical and applied sciences and eager to pass on his knowledge to all levels of the educated community, publishes such a review. The paper is titled *Von der Gestalt der Erden* (On the shape of the Earth) [1] and it appeared in 7 installments, between April 3 and December 25, 1738, in the *Anmerckungen über die Zeitungen* (Notes on the newspapers), a German magazine published in the imperial city. In this article, Euler explains using non-technical words, the reasons and the arguments that led to the two antagonistic theories concerning the figure of the Earth: a prolate or oblate spheroid. In his account, when he talks about the Earth having the form of an oblate spheroid, that is, a spheroid flattened at the Poles and elongated at the Equator, Euler says that it has the shape of an orange. In the contrary case, he says that the Earth has the shape of a melon.

Euler starts by recalling that in the preceding century, supporters of either point of view had strong arguments from physics, and he presents these arguments in simple words. Some of them are based on deep theories involving fluid mechanics and the general natural sciences, and other arguments are based on experimental measurements. Let us note right away that concerning this problem of the figure of the Earth, Euler's most important contribution is purely mathematical; we shall say more on it later in this paper. Euler notes in his *Anmerckungen* article that to address this question from a mathematical point of view, in particular, in order to make use of the methods of differential calculus, one must assume that the surface of the Earth is smooth. For that purpose, he supposes that the surface of the Earth is covered by still water. He notes that if the Earth were melon-shaped, then, bodies would be heavier near the Equator than near the Poles and he claims that the experiments done with pendulums are reliable in order to decide which one among the two prevailing theories is correct. In the case of a melon-shaped Earth, a pendulum beating accurately a second must be longer at the Equator than at the Poles.

Euler then reviews the geometry that lies behind the measurements of degrees of meridians that were carried out, with the help of astronomical observations, near the Poles and at the Equator. He talks about the Lapland and the Peru expeditions whose aim was not only to check whether the Earth is melon or orange-shaped, but also to provide precise measurements for the ratio of the axis of the Earth to its diameter. At the time Euler published his article, the Lapland expedition was already completed and its results indicated that the Earth is orange-shaped, that is, flattened at the Poles and elongated at the Equator. He declares that he is quite confident that the results of the Peru expedition will confirm this fact. Furthermore, the results of the latter expedition will still be important since they will give information on the length of the diameter of the Earth (which should be larger than its axis).

After presenting the problem of the figure of the Earth, Euler writes that he will explain how this question can be solved without appealing to measurements, but by pure reason. He starts by noting that if the Earth were perfectly round, the weight of an object would have the same value on all this surface, and as a vector (that is, represented by the vector of gravity, whose direction is given by a plumb line which indicates the vertical direction at each point), it would be directed towards the center. He notes that the last property is impossible to check practically, but that the fact that an object has the same weight at different locations of the Earth may be checked. In the case of a melon-shaped Earth, gravity near the Equator must be greater than near the Poles, that is, a body is heavier near the Equator than near the Poles. Likewise, in the case of an orange-shaped Earth, the same object would be lighter near the Equator than near the Poles, that is, the force of gravity is greater at the Poles than at the Equator. Euler says that these considerations provide a practical way to check whether the Earth is round, or melon-shaped, or orange-shaped. It is interesting to note that such a gravitational argument was used by Aristotle about 2300 years before Euler, to conclude that the Earth is spherical[4].

Euler then discusses the causes of gravity, and he recalls that this matter is still controversial. He declares that one cannot determine the variation of gravity on the surface of the Earth just by weighing the same object at different places, since the weight used in the scale will become lighter or heavier, like the body itself, and the results will not be conclusive. Therefore, one should study, rather than the weights of bodies, the gravitational force itself, and this can be deduced from the speed of falling bodies. Thus, if near the Equator and the Poles, a body falls from the same height in the same amount of time, then the Earth is spherical. If the height from which a body falls in one second is greater under the Equator than under the Poles, then the gravitational force is greater under the Equator than under the Poles, and in this case the Earth has the shape of a melon. In the contrary case, the Earth has the shape of an orange. But to make such measurements in a very precise way is not an easy matter, and Euler turns to pendulums.

The more the force of gravity is greater, the more the pendulum frequency is greater, for a fixed length. If at two places, pendulums of the same length make the same number of oscillations per unit of time, then the gravitational force is the same at the two places. If this takes place everywhere, then the figure of the Earth is spherical. But if the number of oscillations of a given pendulum is smaller (respectively greater) under the Equator than under the Poles, then the Earth is orange-shaped (respectively melon-shaped). Euler reports on experiments with pendulums that have been carried out at different places of the Earth by means of which it has been shown that the nearer one comes to the Equator, the shorter a pendulum indicating the same time is. Therefore these experiments show that the Earth is flattened at the Poles, that is, it has the form of an orange.

After having shown, with the help of pendulums, that the Earth is thicker at the Equator than near the Poles, the question became that of knowing by how much exactly, i.e., what is the ratio of the diameter of the Equator to the axis of the Earth. For this, Euler explains that one needs to calculate the length of one degree of meridian at different place of the Earth and to compare these measurements. Since the Earth is not spherical, the meridians are ellipses, and the length of degrees of meridians are not equal everywhere.

Another way of studying this question is to compare, near the Equator, the value of a degree of meridian with that of a degree of Equator. Euler recalls that finding the values of these degrees is done using astronomical observations. He explains how the degrees of meridian can be measured and he mentions the observations that were performed in France, and the expeditions to Lapland and Peru. To compute the degrees on the Equator, that is to say, degrees of longitude, it is necessary to observe the eclipses of the Moon, or of the moons of Jupiter.

Euler then talks about the figure of Jupiter, which was also shown to be orange-shaped. Furthermore, astronomical observations have concluded that the axis between the two Poles of this planet is smaller than the diameter of its Equator by a fraction of a tenth. Euler says that if the same thing could be observed for all the other planets, there would be no doubt about the figure of the Earth, and that the mere fact that this flattening of Jupiter was observed already gives us an indication of the possible figure of the Earth. He explains that all this is caused by the action of gravity, exerted on the fluid part that constitutes the Earth.

In fact, Euler was very much involved in questions on gravity. We shall mention below some of his memoirs related to this subject. Furthermore, several passages in his *Letters to a German Princess* are concerned with this subject. Moreover, Euler published in 1743 (anonymously) a memoir on gravity, *De causa gravitatis* (On the cause of gravity) [3],[5] and he took up this subject again in his treatise *Anleitung zur Naturlehre* (Introduction to Natural Science) [4]. For a discussion of Euler's theories of gravity in relation

[4] Aristotle gives this argument in Chapter 14 of Book II of his treatise *On the Heavens*. I discuss this in the first chapter of the volume [2].

[5] It is probable that the reason why Euler remained anonymous is that he wished to avoid the controversies which the question of gravitation drew, between those who adhered to Newton's theories, mainly the English physicists, and those who did not, i.e., the French, German and Swiss. There were exceptions, of course. For example, Christiaan Huygens, the preeminent Dutch mathematician and physicist who lived in France, and Pierre-Louis de Maupertuis, the French astronomer and geographer, who became the president of the Prussian Academy of Sciences while Euler was working there, were supporters of Newton's ideas.

to the ideas of Newton, Descartes and others, I refer the reader to Andreas Kleinert's article [5].

On Euler's other writings in relation with the figure of the Earth

Among Euler's other writings on the figure of the Earth, I start with the geometrical memoir *Élémens de la trigonométrie sphéroïdique tirés de la méthode des plus grands et plus petits* (Elements of spheroidal trigonometry drawn from the method of the maxima and minima) [6] (1755), in which Euler presents the mathematical notions that underly the measurements of degrees of meridians conducted in Peru and Lapland, discussing the possible errors made during these measurements and their impact on the knowledge of the true figure of the Earth. In this memoir, he also studies the general trigonometry of a spheroid.

In order to develop a trigonometrical theory of a curved surface, one needs to introduce the notion of triangle on such a surface, and Euler starts his memoir by defining the notion of geodesic triangle on an arbitrary surface. We recall that spherical triangles were known since the work of Menelaus of Alexandria (1st-2nd c. AD), see [7]. It is possible that we have, in this memoir of Euler, for the first time in history, the notion of triangle on an arbitrary curved surface. After giving this general definition, Euler restricts his study to the case where the surface is a spheroid. The sides of such a triangle are geodesics on the spheroid, and these geodesics are generally, unlike the geodesics on the sphere, doubly curved lines, that is, they are not contained in a plane, a result noticed for the first time by Clairaut in his memoir [8] (1735) of which we shall say more in below.

Let us review some of the mathematical questions that Euler solves in his memoir:

1) Given the latitude of a point on the spheroid, to determine its distance to the centre of the Earth and the osculating radius at this point (Sections 4 and 5 of [6]);

2) Given two points situated on the same meridian, knowing their latitudes, to find the magnitude of the arc of meridian contained between them (Section 17 of [6]);

3) Given two points of which we know the latitudes and the difference between the longitudes, to find the shortest path between them (Section 19 of [6]);

4) To determine the ratio of the diameter of the Equator to the axis of the Earth, without using the measurements done by the expeditions near the Pole and near the Equator, but by a construction done in a small portion the Earth (Section 24 ff. of [6]).

Problem (3) is more general than Problem (2) and its solution needs the full power of the calculus of variations, or the "method of maxima and minima" as Euler used to calls it, a theory that he had developed himself.

To solve Problem (4), Euler proposes a series of astronomical observations, together with the possibility of drawing a straight line (a geodesic) in the given region. Provided this can be done precisely, he gives a formula for the required ratio.

Note that the answers to the first three problems are straightforward in the case of a sphere. In the case of a spheroid, the latitude is defined as the angle made by a perpendicular to the surface of the Earth with its axis of rotation. Note that the perpendicular does not pass through the center of the spheriod, unless the point is on the Equator.

In Section 21 of the memoir, Euler provides a set of formulae which constitute the trigonometry of the spheroid announced in the title of his memoir. These formulae depend on a constant δ which is equal to $\frac{e^2-a^2}{e^2+a^2}$, where e is the semi-diameter and a is the semi-axis of the spheroid. For $\delta = 0$, the formulae give all the known rules of spherical trigonometry which Euler had amply covered in his previous memoirs.

From the practical point of view, Euler, in Sections 12-14 of the same memoir, based on the measurements conducted during the various expeditions, finds that the value of the ratio of the diameter of the Earth to its axis is equal to 230/229, and he notes that this value coincides with the one given by Newton in his *Principia*. He discusses in detail the other ratios that one may find by using other measurements and other methods of calculation. His methods for obtaining these results use approximation techniques. He notes that these methods are advantageous only in regions which are neither too close to the Equator nor to the Poles (§34).

Among the other memoirs of Euler that are concerned with the problems of gravity and the figure of the Earth, I mention the *Methodus viri celeberrimi Leonhardi Euleri determinandi gradus meridiani pariter ac paralleli telluris, secundum mensuram a celeb. de Maupertuis cum sociis institutam* (Method of the celebrated Leonhard Euler for the determination of a degree of a meridian, as well as of a parallel of the Earth, based on the measurement undertaken by the celebrated de Maupertuis and his colleagues) [9] (1750).

The title of this memoir is enough informative. Euler studies there several geographical problems, including the determination of the length of a degree of a meridian at a given latitude, and the determination of the latitude once we know the length of a degree of a parallel.

The subject of the next memoir we consider concerns the influence of the figure of the Earth on astronomy. It was published in 1747, and it is titled *De la parallaxe de la lune tant par rapport à sa hauteur qu'à son azimuth, dans l'hypothèse de la terre sphéroïdique* (On the parallax of the moon, both with respect to its elevation and its azimuth, under the hypothesis of a spheroidal Earth) [10]. The term "parallax" used here refers to the influence of the position of an observer on the trajectory of a celestial object, seen from his own position (the object being, here, the moon). From the mathematical viewpoint, this is a coordinate change problem, in the setting of the geometry of the spheroid. Euler starts his memoir by recalling that Maupertuis published a treatise on the parallax of the moon, in which he showed how the usual rules, that is, under the hypothesis of a spherical Earth, have to be modified, if one takes into consideration the spheroidal shape, but that the latter failed in taking into account one parameter, namely the azimuth, that is, the angle seen from the observer, in a horizontal plane, between the projection of the direction of the celestial object considered, and a given reference direction. He develops in this memoir [10] the trigonometric computations needed in this geometrical problem.

In the memoir *De attractione corporum sphaeroidico-ellipticorum* (On the attraction of spheroido-elliptical bodies) [11], published in 1738, Euler studies gravity on a planet which has a spheroidal shape, made out of a uniform material and in which the particles attract each other by a force whose magnitude is inversely proportional to the squares of the distances and rotates about the axis. Under such hypotheses, he obtains a formula for the attraction law between a particle situated at a Pole and another one at the Equator.

Clairaut

Alexis-Claude Clairaut is another eighteenth century scientist who worked on the problems of the figure of the Earth, both from the physical and mathematical viewpoints. Besides being an outstanding mathematician, Clairaut was an excellent physicist and astronomer. He had a regular correspondence with Euler, and each the two men had a lot of respect for the other's work. They regularly informed each other of their respective works. I have dwelt on the relation between Euler and Clairaut in the article [12].

Like Euler, Clairaut was thoroughly involved in hydrostatics, that is, in the theory of the equilibrium of forces acting on a fluid. His book, *Théorie de la figure de la terre, tirée des principes de l'hydrostatique* (Theory of the figure of the Earth, drawn from the principles of hydrostatics) [13], published in 1743, was motivated by the question of the figure of the Earth, and it constitutes an important piece of work on this subject. He develops there the idea that the Earth, originally constituted by a fluid matter, acquired gradually its spheroidal form, explaining this by the equilibrium laws of hydrostatics. His theory, which confirmed Newton's theory on the same topic, constitutes at the same time an extension of the latter's theory of universal attraction. In this book, we find the so-called Clairaut theorem which says that the Earth is a body in hydrostatic equilibrium under the sum of gravitational and centrifugal potentials satisfying a certain exact differential equation for a homogeneous field. It is also worth mentioning that in this work, Clairaut, while he confirmed Newton's conclusions on the form of the Earth, corrected some of the latter's computations, showing that the ratio of the minor axis to the major axis of the spheroid that it forms is 230/231, which is different from the value that Newton found.

The first part of Clairaut's work [13] is titled *Principes généraux pour trouver les hypothèses dans lesquelles les fluides peuvent être en équilibre, et pour déterminer la figure de la Terre et des autres planètes, lorsque la loi de la pesanteur est donnée.* (General principles for finding the hypotheses under which fluids can be in equilibrium, and for determining the figure of the Earth and of the other planets, given the law of gravity). In §1, Clairaut says that a fluid mass cannot be in equilibrium unless the forces of all parts contained in a channel of an arbitrary figure which we imagine as traversing the entire mass cancel each other. This is the first time that a principle of fluid equilibrium is stated. The methods that Clairaut uses in his work are geometrical. A few years later, with Lagrange's *Mécanique analytique* (1788), analytical methods became dominant in mechanics and hydrostatics.

The Earth as a spheriod was sometimes referred to in the eighteenth century science literature as the *Clairaut spheroid*.

Clairaut, like Euler, wrote several memoirs on the motion of the moon, for whose study he also followed Newton's ideas and in particular his law of attraction, the same law that explains the figure of the Earth. On this subject, we mention his memoir: *De l'orbite de la lune dans le système de M. Newton,* (On the orbit of the moon in the system of Mr. Newton) (1746) [14] and *Du système du monde dans les principes de la gravitation universelle* (On the system of the world in the principles of universal gravitation) (1749) [15].

More directly related to the figure of the Earth are the two memoirs *Détermination géométrique de la perpendiculaire à la méridienne tracée par M. Cassini avec plusieurs méthodes d'en tirer la grandeur et la figure de la terre* (Geometrical determination of the perpendicular to the meridian drawn by Mr. Cassini with several methods of extracting its length and the figure of the Earth) [8] and *Sur la nouvelle méthode de M. Cassini pour connaître la figure de la terre* (On the new method of Mr. Cassini to know the figure of the Earth) [16]. Despite the titles, these memoirs are geometrical. For instance, in [8], Clairaut shows that a geodesic which is not the Equator and which intersects perpendicularly a meridian cannot be planar unless this spheroid is a sphere. Clairaut furthermore determines the nature of that curve.

Motivated by the theory of the form of the Earth, Clairaut worked, before Euler, on the geometry and trigonometry of the ellipsoid of revolution. His two memoirs [8] and [16] contain several theorems on the geodesics on a surface of revolution which is not the sphere, in particular on the curvature of the curves obtained by the intersections of such surfaces with planes, with a special attention to the case of a spheroid.

The memoir *An inquiry concerning the figure of such planets as revolve about an axis, supposing the density continually to vary, from the centre towards the surface* [17] was presented to the Royal Academy of Paris and to the Royal Society of London in 1737–1738, and it was published in the latter's Transactions, translated into English. In this memoir, Clairaut returns to Newton's part of the *Principia* which is concerned with the figure of the Earth. He first announces that some observations he made under the Arctic circle, during the Lapland expedition, led him to believe that this figure was flatter than what Newton thought. and he expresses his surprise concerning the fact that Newton applied different physical theories, as to the causes of this ellipticity, regarding the Earth and Jupiter. But Clairaut is mostly interested in geometry, and the core of his memoir is mathematical. Among the problems that he discusses, we mention the following three:

- *Problem 1: To find the attraction which a homogeneous spheroid, differing but very little from a sphere, exerts upon a corpuscle placed at a point on the axis of revolution.*
- *Problem 2: The spheroid is no more supposed to be of a homogeneous matter, but composed of an infinite number of ellipsoidal strata which are all similar, and whose densities are represented by an arbitrary curve whose equation is known. To find the attraction that it exerts on a corpuscle placed at a Pole.*
- *Problem 3: To find the attraction which a spheroid exerts upon a corpuscle placed at an arbitrary point of its surface.*

From the point of view of fluid mechanics, we quote Lagrange, who declares in his *Mécanique analytique* that Clairaut changed the face of Hydrostatics, and made it a new science. He writes [18, t. 1, p. 179–180]:

> [. . .] Clairaut made [Newton's principle] more general, by showing that the equilibrium of a fluid mass requires that the forces of all the components of the fluid enclosed in an arbitrary channel, ending at the surface or entering into itself, destroy each other. He was the first to deduce, from this principle, the true fundamental laws of equilibrium of a fluid mass whose parts are animated by arbitrary forces, and he found the partial difference equations by which these laws can be expressed, a discovery which changed the face of Hydrostatics, which he made a new science.

After this passage, Lagrange talks about Euler's work on hydrostatics, which, he says, is adopted in almost all the treatises on this science.

Lagrange was, among the 18th century preeminent mathematicians, Euler's young competitor who was closest to him in terms of depth of thought. We shall talk about him in the next section.

Lagrange

Motivated by the question of the figure of the Earth, Lagrange, in 1773, published a memoir titled *Sur l'attraction des sphéroïdes elliptiques* (On the attraction of elliptical spheroids) [19] in which he computed the attraction force exerted by an ellipsoid on a point situated either in the interior or outside this ellipsoid, but in the latter case the point is assumed to be on situated on one of its axes. In fact, Lagrange worked out this problem for an ellipsoid with three axes of different lengths, that is, an ellipsoid which is more general than a spheroid. In the memoir we just mentioned, he recovers formulae that were established by Colin Maclaurin on the same question, recalling that the latter has already solved this question in a memoir which won a prize by the French Academy of Sciences in 1740. MacLaurin used geometric methods, and Lagrange says that from this point of view, this memoir is comparable to the most beautiful memoirs of Archimedes. But at the same time Lagrange finds it useful to have a solution that is based on analysis (differential and integral calculus), and this is the object of his own memoir.[6] He declares in his memoir [19] that his aim, in giving a solution by analysis of the questions considered by Maclaurin, "will serve to destroy one of the main arguments that the detractors of Analysis can bring to lower it and to prove the superiority of the synthetic method of the Ancients".

Among the problems that Lagrange considers, we mention the following three:

• *To find the general expression of the attraction that a body of a given figure exerts on a point placed wherever one wishes, assuming that each particle of the body attracts this point as an arbitrary function of distance.*

• *To determine the value of the attraction that a body whose surface is expressed by a second degree equation exerts on a point placed inside the body or on its surface, assuming the attraction is reciprocally proportional to the squares of the distances.*

• *Under the assumptions of the preceding problem, to find the attraction exerted on the a point placed outside the body.*

Legendre: geodesy and approximation

Legendre is one of the preeminent mathematicians of the 18th-19th centuries who were heavily involved in geodesy. In 1787, he collaborated with Jean-Dominique Cassini de Thury (also known as Cassini IV) for the determination of distances on the coasts of France and England using the method of triangulations. In his *Mémoire sur les opérations trigonométriques, dont les résultats dépendent de la figure de la terre* (Memoir on the trigonometric operations whose results depend on the figure of the Earth) [20], published the same year, he gives some formulae that are necessary for geodesic calculations in triangles on the surface of a spheroid whose curvature is "infinitely small", i.e., triangles whose sides are very small compared to the radius of the Earth. Such triangles appear in the method of triangulations used in geodesy. The formulae give what Legendre calls the angle at the horizon, the angle of depression or elevation of an observed point relative to the horizon at the place of observation, the distance to the horizon, the excess over 180° of the sum of the three angles of a reduced triangle at the horizon, the value of a degrees of meridian on the spheroid, as well as distances relative to the North Star. He talks about the use of the so-called *repeating circle*, an instrument introduced in France by Charles de Borda and Étienne Lenoir at the end of the 18th century for measuring astronomical distances for the purpose of geodesy. In the same memoir, Legendre obtains simple formulae for the shortest line which starts at a given point and making with the meridian a given angle. He writes that his work is motivated by problems in geography, in particular, those of establishing precise measurements for the coast of France.

In his memoir *Analyse des triangles tracés sur la surface d'un sphéroïde* (Analysis of triangles drawn on the surface of a spheroid) [21], read to the Academy of Sciences in 1806, Legendre continues the study of the geometry of the spheroid for its practical use in geodesy, and more precisely, for the purpose of showing the exactness of the measures that he conducted for the computation of distances between Dunkerque and Montjouy, near Barcelona, using a chain of geodesic triangles. From the mathematical point of view, Legendre considers the following problem: On the surface of the spheroid whose axis and diameter are known, given a triangle such that we know the following five quantities: two sides and a vertex they contain, the latitude of this vertex and the azimuth of one side, to determine the third side of the triangle (and therefore, to determine completely the triangle). He gives an approximate value of this side up to order 4, under the hypothesis that the Earth flattening is small. More precisely, he obtains a good approximation of a quantity z such that if the value of the known angle is lowered by this quantity, the third side is equal to the one of a Euclidean triangle formed by the two known sides and the angle $A - z$ they contain. At the same time, he shows that the difference in angles between this spheroidal triangle and the spherical

[6] As a matter of fact, it is known that Lagrange, in several works (and the calculus of variations is one characteristic example), replaced Euler's geometric arguments by analytic ones.

triangle whose sides have the same lengths is of order three, and he notes that for the work in geodesy, the differences between such triangles are negligible.

It is interesting to see that Felix Klein, in vol. III of his famous treatise *Elementary mathematics from a higher standpoint* [22], comments on Legendre's approximations in geodesy (see p. 175), and in particular on a theorem of the latter extracted from his *Géométrie* [23, p. 426] which says the following:

Suppose we are given a small triangle of angles α,β,γ *on a sphere of radius one and a plane triangle of the same side lengths and angles* α',β',γ'. *Let us set* ϵ *to be the angle excess of the spherical triangle, that is,* $\epsilon = \alpha + \beta + \gamma - 180^o$. *Then we have* $\alpha' = \alpha - \frac{\epsilon}{3}$, $\beta' = \beta - \frac{\epsilon}{3}$, $\gamma' = \gamma - \frac{\epsilon}{3}$.

Such a formula allows to conduct geodesic measurements with Euclidean triangles rather than spherical, which is much simpler for numerical calculations.

The same work by Klein contains a chapter on geodesy and on the measurements of gravity, under the hypothesis that the Earth is a spheroid.

Talking about Legendre, we finally mention his article with Delambre titled *Méthodes analytiques pour la Détermination d'un arc du Méridien* (Analytical methods for determination an arc of the meridian) [24] (1798).

Gauss, mathematician and geographer

C.-F. Gauss generalized Legendre's considerations from the theoretical and applied point of view, again with practical surveying goals in mind, see [25, p. 115]. Dombrowski, the modern editor of Gauss's *Disquisitiones generales circa superficies curvas* [25], quotes a letter from the latter to his friend H. W. Olbers in which he writes: "[. . .] In practice this [the difference of the correction values for the different angles of terrestrial geodesic triangles] is of course not at all important, because it is negligible for the largest triangles on Earth that can be measured; however, the dignity of science requires that we understand clearly the nature of this inequality".

We recall that Gauss, besides being arguably the most important mathematician of his time, was also a geographer and a geodesist. Being the head of the University of Göttingen's observatory, geography was part of his official activities. His works on the differential geometry of surfaces was motivated by questions on geography. In 1820, he was in charge of the project of measuring the extent of the Kingdom of Hanover, and for this, he realized a triangulation of these regions. His famous paper titled *Allgemeine Auflösung der Aufgabe: die Theile einer gegebnen Fläche auf einer andern gegebnen Fläche so abzubilden, daß die Abbildung dem Abgebildeten in den kleinsten Theilen ähnlich wird* (General solution of the problem: to represent the parts of a given surface on another so that the smallest parts of the representation shall be similar to the corresponding parts of the surface represented) [26], published in 1825, which contains his famous results on the conformal representation of simply connected surfaces, was motivated by geographical questions. Indeed, he writes in this paper that his aim is only to construct geographical maps and to study the general principles of geodesy for the task of land surveying.

Gauss's famous geometrical paper, the *Disquisitiones generales circa superficies curvas* (General investigations on curved surfaces), which contains his *Theorema Egregium* saying that curvature is the only obstruction for a surface to be faithfully represented on the plane (§12; p. 20 of the English translation), contains precise measurements of angles between Mt. Hohehagen, Brocken and Inselberg. We also mention Gauss's 1828 paper on the spheroid, titled *Conforme Abbildung des Sphäroids in der Ebene (Projectionsmethode der Hannoverschen Landesvermessung)* (Conformal mapping of the spheroid in the plane; projection method of the Hannoversche Landesvermessung) [27].

Jacobi and the relation with Abelian integrals

In the 19th century, the study of geodesics on the spheroid, and more generally, on the ellipsoid, was an important research topic, and several mathematicians worked on it. C. G. J. Jacobi studied this problem in several memoirs, see e.g. [28, 29] and the memoir published in 1841 and titled *De la ligne géodésique sur un ellipsoïde, et des différents usages d'une transformation analytique remarquable* (On the geodesic line on an ellipsoid and the various usages of a remarkable analytic transformation) [30]. In the last memoir, Jacobi declares that his motivation for the study of this problem arises from geography, and he mentions works of Lambert and Gauss on this topic, as well as works by Euler on mechanics. Jacobi was led in this study to abelian integrals, which is one of his favorite subjects. He studied similar problems of geodesy using elliptic functions.

In another memoir published in 1857 and titled *Solution nouvelle d'un problème fondamental de géodésie* (A new solution of a fundamental problem in geodesy) [31], Jacobi considers, on an ellipsoid having the shape of the Earth, a geodesic arc whose length is known, as well as the latitude and the azimuth angle at its origin, and he studies the question of finding the latitude and the azimuth angle at the extremity of this arc, as well as the difference in longitudes between the origin and the extremity. He writes that the same problem has been recently treated with particular care by Gauss, who gave different solutions of it.

Jacobi also included the study of closed geodesics on the ellipsoid in the context of dynamics, see [32]. Weierstrass continued Jacobi's work on closed geodesics on an ellipsoid and he introduced in this study theta functions, see [33].

Other works

Among the other works on the geometry of the spheroid motivated by questions of geography, let me mention Laplace's *Mécanique céleste* (Celestial mechanics, first version 1798) [34, p. 128ff], in which the latter studies the curves on a spheroid whose length is shortest among the curves joining two given points.

J. D. Gergonne wrote an article on loxodromic curves on an ellipsoid, *De la loxodromie, sur une surface de révolution, et, en particulier, sur un sphéroïde elliptique* (On loxodromy, on a surface of revolution, and in particular, on an elliptic spheroid) [35] (1817–1818). Here, the word *loxodromy* denotes a line on a surface of revolution which makes a constant angle with meridians. The author establishes a differential equation satisfied by such a curve and he solves it in the case of a spheroid.

Among the other mathematicians of the same period who were also geographers, we mention F. W. Bessel, who, in his paper *Über die Berechnung der geographischen Längen und Breiten aus geodätischen Vermessungen* (The calculation of longitude and latitude from geodesic measurements) [36], gave a solution to the problem of the geodesics on the spheroid for its use in geography. An English translation of this paper by Ch. F. F. Karney and R. E. Deakin, is available. The same author, in 1837, wrote a paper titled *Bestimmung der Axen des elliptischen Rotationssphäroids, welches den vorhandenen Messungen von Meridianbögen der Erde am meisten entspricht* (Determination of the axes of the elliptical rotational spheroid that is most consistent with existing measurements of Earth meridian arcs).

At the turn of the twentieth century, Poincaré studied geodesics on spheroids in the context of the theory of differential equations, more precisely, using a method of Lagrange that the latter introduced in his work on the motion of a planet under the action of perturbations due to other planets, and which he calls the "theory of variation of the constants". In Poincaré's terminology, a *spheroid* is a convex surface which is sufficiently close to a sphere. In some sense, this study is a generalization of a study that Poincaré made earlier of geodesics on the sphere. In his paper *Sur les lignes géodésiques des surfaces convexes* (On the geodesic lines of convex surfaces) [37], published in 1905, Poincaré is interested in the number of stable closed geodesics on such a surface, that is, closed geodesics that remain so under small deformations. He shows that in the case of the spheroid, this number is odd. After treating this question on a spheroid, he addresses the same question in the case of a general convex surface. Poincaré's interest in this kind of questions is motivated by their relation with the 3-body problem.

Many other works on the geometry of the spheroid, or the ellipsoid, were done in the 19th century by various authors. We mention works of Cayley [38], Gundermann [39], Forsyth [40, 41] Helmert [42], Ivory [43], Kummell [44], Krüger [45], Liouville [46–48], Monge [49], Oriani [50, 51] Puissant [52, 53] and Stein [54].

We end our list with a more recent article. L. E. Ward, in a paper titled *Geodesics and plane arcs on an oblate spheroid* published in 1943 [55], studies the following question:

Given two points P_1 and P_2 on a spheroid that are not on the same meridian, let s_{12} be the length of the geodesic connecting them, and σ_{12} be the length of arc which is the intersection of the spheroid with the plane containing P_1 and P_2 and which passes by the center of the spheroid. By how much does σ_{12} exceed s_{12}?

The author declares that this question is motivated by the current interest in navigation.

Later in the 20th century, J. Moser studied the geodesics on the spheroid in the context of isospectral deformations of surfaces, see [56, 57].

Conclusion

The geometry of the spheroid was already studied by Archimedes, back in the 3rd century BC (see the *Method treating of mechanical problems*, a work that Archimedes dedicated to Eratosthenes, who was, among other attributes, a major geographer [58]). Of course, neither Archimedes nor Eratosthenes talked about the Earth being a spheroid. It was after the geographical discoveries of the eighteenth century concerning the true figure of the Earth that the study of the spheroid was carried out in the setting of differential geometry and the calculus of variations.

It is interesting to see how questions in geography contributed to the development of geometry.

REFERENCES

1. Euler L. Von der Gestalt der Erden, Anmerckungen über die Zeitungen, St. Petersburg 3 April 1738 – 25 December 1738. *Opera Omnia*. Ser. III, vol. 2, p. 325–346.
2. Caddeo R., Papadopoulos A. *Geography in the Eighteenth century: Euler, Lagrange and Lambert*. Springer Nature, 2022.
3. Euler L. De causa gravitatis. Miscellanea Berolinensia. *Euler*. 1743;7(31):360–370.
4. Euler L. Anleitung zur Naturlehre. *Opera Postuma*. 1862;2:449–560. *Opera Omnia*. Series 3, Volume 1, p. 16–180.
5. Kleinert A. Euler, Lesage et Newton: Trois approches différentes pour résoudre le problème de la pesanteur, In: X. Hascher et A. Papadopoulos (ed.). *Leonhard Euler, mathématicien, physicien et théoricien de la musique*. CNRS Éditions, Paris, 2015.
6. Euler L. Élémens de la trigonométrie sphéroïdique tirés de la méthode des plus grands et plus petits, *Mémoires de l'académie des sciences de Berlin*. 1755;9:258–293. *Opera Omnia*. Series 1, Volume 27, p. 309–339. English translation in the present volume.
7. Rashed R., Papadopoulos A. *Menelaus' Spherics: Early Translation and al-Māhānī/al-Harawī's Version*. De Gruyter, Series: Scientia Graeco-Arabica, 21, 2017. 890 p.
8. Clairaut A.-C. Détermination géométrique de la perpendiculaire à la méridienne tracée par M. Cassini avec plusieurs méthodes d'en tirer la grandeur et la figure de la terre. *Mém. de l'Acad. Roy. des Sciences de Paris*. 1735, p. 404–416.
9. Euler L. Methodus viri celeberrimi Leonhardi Euleri determinandi gradus meridiani pariter ac paralleli telluris, secundum mensuram a celeb. de Maupertuis cum sociis institutam. *Commentarii academiae scientiarum Petropolitanae*. 1750;12:224–231. *Opera Omnia*. Series 2, Volume 30, p. 73–88.
10. Euler L. De la parallaxe de la lune tant par rapport à sa hauteur qu'à son azimuth, dans l'hypothèse de la terre sphéroïdique. *Mémoires de l'académie des sciences de Berlin*. 1751;5:326–338. *Opera Omnia*. Series 2, Volume 30, p. 140–150.
11. Euler L. De attractione corporum sphaeroidico-ellipticorum. *Commentarii academiae scientiarum Petropolitanae*. Volume 10, p. 102–115. *Opera Omnia*. Series 2, Volume 6, pp. 175–188.
12. Papadopoulos A. Clairaut, Euler and the figure of the Earth. *Ganita-Bharati*: *Bulletin of the Indian Society for History of Mathematics*. 2022.
13. Clairaut A. *Théorie de la figure de la terre, tirée des principes de l'hydrostatique*. Paris, David Fils, 1743.
14. Clairaut A.-C. De l'orbite de la lune dans le système de M. Newton. *Mém. Acad. Roy. Sc*. 1746, p. 17–32.
15. Clairaut A.-C. Du système du monde dans les principes de la gravitation universelle. *Mém. Acad. Roy. Sc*. 1749, p. 329–364.
16. Clairaut A.-C. Sur la nouvelle méthode de M. Cassini pour connaître la figure de la terre. *Mém. de l'Acad. Roy. des Sciences de Paris*. 1738, p. 117–122.
17. Clairaut A.-C. An Inquiry concerning the Figure of Such Planets as Revolve about an Axis, Supposing the Density Continually to Vary, from the Centre towards the Surface, Translated from the French by the Rev. John Colson Lucas. *Philosophical Transactions*. 1737–1738;40:277–306.
18. Lagrange J.-L. *Mécanique analytique*. Paris: Nouvelle édition, Courcier; 1811.
19. Lagrange J.-L. Sur l'attraction des sphéroïdes elliptiques. *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin*. 1773. Œ uvres, vol. 3, p. 619–658.
20. Legendre A.-M. Mémoire sur les opérations trigonométriques, dont les résultats dépendent de la figure de la terre. *Mémoires de l'Académie des Sciences de Paris*. 1787, p. 252–283.

21. Legendre A.-M. Analyse des triangles tracés sur la surface d'un sphéroïde, Lu à la séance de la classe des sciences mathématiques et physiques de l'Institut, le 3 mars 1806. *Mémoires de la classe des sciences mathématiques et physiques de l'Institut national de France*. Ier sem., p. 130–161.
22. Klein F. *Elementary mathematics from a higher standpoint, Volume III: Precision mathematics and approximation mathematics*. Transl. by M. Menghini and A. Baccaglini-Frank, 2016, Springer.
23. Legendre A.-M. *Géométrie*. 12e édition, Paris, 1823.
24. Delambre J.-B.-J. *Méthodes analytiques pour la détermination d'un arc du méridien, Précédées d'un mémoire sur le même sujet par A.-M. Legendre*. Paris, Duprat, 1798.
25. Dombrowski P. 150 years after Gauss' "disquisitiones generales circa superficies curvas". *Astérisque*. 1979;62.
26. Gauss C. F. Allgemeine Auflösung der Aufgabe: die Theile einer gegebnen Fläche auf einer andern gegebnen Fläche so abzubilden, daß die Abbildung dem Abgebildeten in den kleinsten Theilen ähnlich wird. *Astronom. Abh.* (Schumacher ed.) 1825;3:1–30 (Also in *Gauss's Werke*. 4:189–216).
27. Gauss C. F. Conforme Abbildung des Sphäroids in der Ebene (Projectionsmethode der Hannoverschen Landesvermessung), König. Ges. Wiss., Göttingen (1828). *Carl Friedrich Gauss Werke*. (Ges. Wiss., Göttingen) 1903;9:142–194.
28. Jacobi C. G. J. Lettre de M. Jacobi à M. Arago concernant les lignes géodésiques tracées sur un ellipsoide à trois axes. *Comptes Rendus des Séances de l'Académie des Sciences*. Paris. 1839;8:284.
29. Jacobi C. G. J. Note von der geodätischen Linie auf einem Ellipsoid und den verschinden Anwendungen. *Journal für die reine und angewandte Mathematik*. 1939;19:309–313.
30. Jacobi C. G. J. De la ligne géodésique sur un ellipsoïde, et des différents usages d'une transformation analytique remarquable. *Journal de mathématiques pures et appliquées 1re série*. 1841;6:267–272.
31. Jacobi C. G. J. Solution nouvelle d'un problème fondamental de géodésie. *Journal für die reine und angewandte Mathematik*. 53:335–341. *Gesammelte Werke*. 1857;2:417–424.
32. Jacobi C. G. J. *Vorlesungen über Dynamic. Gesammelte Werke*. Supplement band, Berlin, 1884.
33. Weierstrass K. Über die geodätischen Linien dem dreiachsigen Ellipsoid. *Monatsber. Königl. Akad. Wiss*. 1861, p. 257–273.
34. De Laplace P. S. *Mécanique céleste*. Vol. 2. Œ uvres. Vol. 2, Paris: Imprimerie Royale; 1843.
35. Gergonne J. D. De la loxodromie, sur une surface de révolution, et, en particulier, sur un sphéroïde elliptique. *Annales de Mathématiques pures et appliquées*. 1817–1818;8:125–136.
36. Bessel F. W. Über die Berechnung der geographischen Längen und Breiten aus geodätischen Vermessungen (The calculation of longitude and latitude from geodesic measurements). *Astronomische Nachrichten*. 1825;4(86):241–254.
37. Poincaré H. Sur les lignes géodésiques des surfaces convexes. *Transactions of the American Mathematical Society*. 1905;6(3):237–274.
38. Cayley A. On the geodesic lines on an ellipsoid. *Mem. Roy. Astron. Soc*. 1872;39:31–53 & plate 2. *The Collected Mathematical Papers of Arthur Cayley*. 1894;7 (Cambridge Univ. Press), paper 478, pp. 493–510.
39. Gudermann C. Fundamenta trigonometriae sphaeroïdicae exacta; imprimis de lineis brevissimis, vulgo dictis geodaeticis, in superficie sphaeroïdica. *Jour. Crelle*. 1852;43:294–339.
40. Forsyth A. R. Geodesics on an oblate spheroid. *Mess. Math*. 1896;25:81–124.
41. Forsyth A. R. Conjugate points of geodesics on an oblate spheroid. *Mess. Math*. 1896;25:161–169.
42. Helmert F. R. Entwicklung der ersten Glieder für die Reduction eines sphäroidischen Dreiecks auf ein sphärisches mit denselben Seiten. *Z. f. Vermess*. 1889;18(9):257–268.
43. Ivory J. On the properties of a line of shortest distance traced on the surface of an oblate spheroid. *Phil. Mag*. 1826;67(336):241–249. 1826;67(337):340–352.
44. Kummell C. H. On the determination of the shortest distance between two points on a spheroid. *Astron. Nachr*. 1885;112:97–108.
45. Krüger J. H. L. *Die geodätische Linie des Sphäroids und Untersuchung darüber, wann dieselbe aufhört, kürzeste Linie zu sein*. Inaugural-Dissertation, Univ. Tübingen, Schade, Berlin, 1883.
46. Liouville J. Démonstration géométrique relative à l'équation des lignes géodésiques sur un ellipsoïde quelconque. *Comptes Rendus des Séances de l'Académie des Sciences*. Paris, 22. 1846, p. 111–113.
47. Liouville J. Théorèmes concernant les lignes géodésiques et les lignes de courbure de l'ellipsoïde.

Comptes Rendus des Séances de l'Académie des Sciences. Paris, 22. 1846, p. 893.

48. Liouville J. De la ligne géodésique sur un ellipsoïde quelconque. *Journal de Mathématiques Pures et Appliquées*. Paris, 9. 1844, p. 401–408.
49. Monge G. Sur les lignes de courbure de la surface de l'ellipsoïde (1796). *Application de l'Analyse à la Géométrie*. 5th edition, Bachelier, Paris, 1850, p. 139–160.
50. Oriani B. *Elementi di trigonometria sferoidica*, Part 1: Mem. dell'Ist. Naz. Ital. Part 1: 1(1), 118–198 (Bologna, 1806); Part 2: 2(1), 1–58 (Bologna, 1808); Part 3: 2(2), 1–58 (Bologna, 1810); Addendum: Mem. dell'Imp. Reg. Ist. del Regno Lombardo-Veneto 4, 325–331 (Milan, 1833).
51. Oriani B. Essempi di calcolo nella soluzione di alcuni problemi di trigonometria sferoidica. *Effemeridi astronomiche di Milano*. 1827, 3–24 (1826); 1828, 3–32 (1827); 1829, 3–24 (Milano 1828).
52. Puissant L. Note sur la trigonométrie sphéroïde, dans laquelle on détermine généralement la plus courte distance de deux point donnés sur la terre par leur latitude et leur longitude Conn. des Tems, 1832, 34–48 (Bachelier, Paris, 1829).
53. Puissant L. Nouvel essai de trigonométrie sphéroïdique. *Mém. l'Acad. Roy. des Sciences de l'Inst. de France*. 1831;10:457–529.
54. Stein J. P. W. *Geographische Trigonometrie oder die Auflösung der geradlinigen, sphärischen und sphäroidischen Dreiecke*. Mainz, 1825. Part 2, Chap. 2.
55. Ward L. E. Geodesics and plane arcs on an oblate spheroid. *American Math. Monthly*. 1943;50(7):423–429.
56. Moser J. Various aspects of integrable Hamiltonian systems. *Dynamical Systems*. 1980. *Progress in Mathematics*, 8, Birkhäuser, p. 233–287.
57. Moser J. Geometry of quadrics and spectral theory. *The Chern Symposium*. 1979, Springer Verlag, 1980, p. 147–188.
58. Heath T. L. *The works of Archimedes*. Cambridge: at the University Press, 1897, reprint, Dover, 2002.
59. Caddeo R., Hascher X., Jedrzejewski F., Papadopoulos A. *Christiaan Huygens: Écrits sur la musique et le son, critical edition, with mathematical, musical and historical commentaries*. Paris: Hermann (coll. Théorie de la musique); 2021, 496 p.
60. Gauss C. F. Conforme Abbildung des Sphäroids in der Ebene (Projectionsmethode der Hannoverschen Landesvermessung). König. Ges. Wiss., Göttingen (1828). *Carl Friedrich Gauss Werke*. (Ges. Wiss., Göttingen) 1903;9:142–194.
61. Gauss C. F. *Disquisitiones generales circa superficies curvas, English translation: General investigations of curved surfaces of 1825 and 1827*. Translated with notes and a bibliography by J.C. Morehead and A.M. Hilterbeitel. Princeton, 1902.
62. De Laplace P. S. Mémoire sur la figure de la terre. *Mémoires de l'Académie Royale des Sciences de Paris*. 1786, p. 17–47. *Œ uvres complètes*. Vol. 11.