DOI: 10.51790/2712-9942-2021-2-2-4

ON EULER AND CHEBYSHEV, AT THE FOUNDATION OF RUSSIAN MATHEMATICS

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Abstract: on the occasion of Chebyshev's two-hundredth anniversary, I review part of his work, showing that in several respects he was the heir of Euler. In doing this, I consider the works of Euler and Chebyshev on three topics in applied science: industrial machines, ballistics and geography, and then on three topics in pure mathematics: integration, continued fractions and number theory, showing that in each filed the two mathematicians were interested in the same kind of questions.

Keywords: Euler, Chebyshev, Russian mathematics, Segner machine, steam engine, geography, artillery, ballistics, gunnery, cartography, elliptic integrals, continued fractions, distribution of primes.

Cite this article: Papadopoulos A. On Euler and Chebyshev, at the Foundation of Russian Mathematics. *Russian Journal of Cybernetics*. 2021;2(2):42–63. DOI: 10.51790/2712-9942-2021-2-2-4.

ПРО ЭЙЛЕРА И ЧЕБЫШЁВА, ОСНОВАТЕЛЕЙ РОССИЙСКОЙ МАТЕМАТИКИ

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Аннотация: по случаю двухсотлетнего юбилея Чебышёва я изучил часть его работ и пришел к выводу, что в целом ряде моментов он продолжает научные традиции Эйлера. Я рассматривал работы Эйлера и Чебышёва по трем прикладным темам: промышленное оборудование, баллистика и география, а также по трем разделам чистой математики: интегральное исчисление, непрерывные дроби и теория чисел. Показано, что во всех случаях оба математика изучали один и тот же круг вопросов.

Ключевые слова: Эйлер, Чебышёв, российская математика, Сегнерово колесо, паровая машина, география, артиллерия, баллистика, оружейное дело, картография, эллиптические интегралы, непрерывные дроби, распределение простых чисел.

Для цитирования: Пападопулос А. Про Эйлера и Чебышёва, основателей российской математики. *Успехи кибернетики*. 2021;2(2):42–63. DOI: 10.51790/2712-9942-2021-2-2-4.

Introduction

We celebrate this year the two-hundredth anniversary of the birth of Pafnuty Lvovich Chebyshev (1821–1874), considered as the founder of a remarkable mathematical school in Saint Petersburg usually referred to as the "first Russian mathematical school", to distinguish it from the previous mathematical school, which can naturally be termed "first mathematical school in Saint Petersburg", that prevailed there for more a century, which was founded by Leonhard Euler and which was dominated by the latter's disciples and by mathematicians that were mostly trained in the West.

Chebyshev and Euler were both involved in several fields of pure and applied mathematics. For each of them, a non-negligible part of the work he did in theoretical mathematics was impelled by applications in the other sciences, and, conversely, the applications he worked on were steered by his research in pure mathematics. Since it is not possible in a few pages to consider all the subjects on which these two prominent mathematicians worked, I have chosen, to illustrate my argument, six topics: three from pure mathematics and three from applied. I will start with the latter. In the next sections, I will talk respectively about machines, artillery, geography, then integration, continued fractions and number theory. I have already considered relations between Euler and Chebyshev's works in the article [1] which appeared in the same journal.

It is known that Chebyshev was a devoted reader of Euler, and more generally, of mathematicians from the past. The authors of the biography contained in his Collected Works edition [2] write that he was an industrious reader of Euler, Lagrange, Gauss and Abel, and that as a general rule he avoided reading the mathematical works of his contemporaries, considering that this would distract him from having original ideas. Thus, it is not surprising that Chebyshev was influenced by the work of Euler. I will take advantage of this commemoration to talk about the two mathematicians, Euler and Chebyshev, highlighting several ways in which the latter was a heir of the former.

Most of Euler's written work and correspondence is now published in his *Opera Omnia* (Collected Works edition), and I shall refer to it. When quoting Chebyshev's works, I shall generally use his French Collected works edition [2], and sometimes the more recent Russian Collected works edition [3] which contains more material.

The present text arose from notes I wrote for three talks I gave at conferences dedicated to the 200th anniversary of Chebyshev: on May 14, 2021, in Obninsk, the chief-town of a small region to which belongs Okatovo, the village where Chebyshev was born; on May 16, 2021 at the Euler International Mathematical Institute in Saint Petersburg, and on May 23, 2021 at Surgut State University. I would like to thank Valerii Galkin, the organizer of the conferences in Obninsk and Surgut, for his invitation to give the talks there, and for motivating me to write this article, and Anton Baranov, who invited me to give the talk at the Saint Petersburg conference.

Mechanical engines

Let me start by a few words of historical context, which will bring us back to the foundation of the city of Saint Petersburg.

Peter the Great's contribution to the development of mathematics in Russia is seldom enough emphasized. Mechanical engines, which constitute the subject of the present section, together with the sciences of artillery and geography which we shall discuss later in this paper, were among his priorities when he decided to found the Saint Petersburg Academy of Sciences. At the same time, the tsar was aware of the fact that in order for his country to be at the forefront in these fields, it was necessary to promote there the study of mathematics.

Bringing mathematics to Russia was not an easy task. Peter the Great first attempted to send Russians to learn mathematics in Western Europe, but this was not conclusive, because the adaptation of the Russian young men that were sent turned out to be difficult. He then created what was called "ciphering" schools for teaching elementary mathematics to Russian children, but this also did not bring significant results. He ordered the translation of mathematical works of Euclid, Descartes, Leibniz and Newton, but the impact of this project was also low, because of lack of readers. The impetus for the beginning of high-level mathematics in Russia came later, after the tsar made several visits to the West, where he encountered preeminent scientists, in particular at the Paris Royal Academy of Sciences. He decided to attract outstanding mathematicians to Saint Petersburg, in order to establish a strong mathematics section at the Academy of Sciences which he had decided to found, and which he wanted to be of the same level of the French Academy of Sciences or the London Royal Society, with the additional stipulation that the scientists in the Saint Petersburg Academy were asked to teach. This is how remarkable mathematicians like Nicolas and Daniel Bernoulli and Christian Goldbach arrived to Saint Petersburg, followed by the young Leonhard Euler (he was 19), who became later on the leading mathematician of the eighteenth century. We refer the reader to the book [4] for a comprehensive account of this side of Russian science history.

Euler was interested in all fields of pure and applied mathematics, and mechanics was one of his favorite subjects. His work on this topic transformed it into a mathematical science, based on integral and differential calculus. He introduced methodically in the solution of mechanical problems the techniques he developed for differential equations, integration and infinite series. Among his achievements in mechanics, we mention the systematic use of the equations of motion, a thorough study of the dynamics of rigid bodies, and the application of the methods of the calculus of variations in the solution of problems in elasticity and hydrodynamics. A major reference for Euler's work on mechanics is his 2-volumes *Mechanica* [5, 6], but he also wrote many other memoirs on this topic, and we shall review some of them.

Besides theoretical mechanics, Euler was interested in several practical mechanical problems, making connections between mathematics and engineering. He wrote more than 40 memoirs on machines. We mention a few of them, because they give us an idea of the questions he dealt with:

- *Discussion plus particulière de diverses manières d'élever de l'eau par le moyen des pompes avec le plus grand avantage* (A more particular discussion on the various methods of raising water using pumps with the greatest advantage) [7] (1754);

- *Recherches plus exactes sur l'effet des moulins à vent* (More exact researches on the effect of windmills) [8] (1758);
- *Sur l'action des scies* (On the action of saws) [9] (1758);
- *Scientia navalis* (Naval science), a fundamental treatise in 2 volumes totaling more than 1000 pages, published in 1738 and 1749, [10, 11], in which Euler sets the foundations of the theory of naval science; the treatise includes sections on hydrostatics, equilibrium, stability, resistance, etc.

To give an idea of some of Euler's work on machines, I will consider especially his memoirs in which he studied a turbine known as the Segner turbine (also called Segner wheel). This is a water turbine which was used, in the eighteenth century, as a low cost power source for mills, in Germany and probably in other countries as well. The machine was conceived around the beginning of the 1750s by the Hungarian-born mathematician and engineer Johann Andreas von Segner (1704–1777), who was the first professor of mathematics at the University of Göttingen and who, later, at the recommendation of Euler, became professor in Halle and corresponding member of the Saint Petersburg and Berlin Academies and of the London Royal Society. His discovery of the turbine that was later called after his name was based on work on water jet action done by Daniel Bernoulli, who was Euler's close friend and among his first colleagues in Saint Petersburg. Segner's machine is also considered as a modern version of the aeolipile, a water turbine invented by Heron of Alexandria (1st c. AD) which works according to the same principle and which is considered as the first known steam engine.

During his stay in Berlin (1741–1766), Euler became thoroughly interested in the Segner turbine, and he wrote several memoirs on this topic, among which we mention the following:

- *Recherches sur l'effet d'un machine hydraulique proposée par M. Segner, professeur à Göttingue* (Researches on the effect of a hydraulic machine proposed by M. Segner, professor at Göttingen), [12], published in 1752;
- *Application de la machine hydraulique de M. Segner à toutes sortes d'ouvrages et de ses avantages sur les autres machines hydrauliques dont on se sert ordinairement* (Application of the hydraulic machine of Mr. Segner to all sorts of situations, and on its advantages over the other hydraulic machines which are ordinarily used) [13], published in 1753;
- *Théorie plus complète des machines qui sont mises en mouvement par la réaction de l'eau* (A more complete theory of the machines that are put in motion by water reaction) [14], published in 1756;
- *De motu et reactione aquae per tubos mobiles transfluentis* (On the motion and reaction of water flowing through moving tubes) [15], published in 1761;
- *Détermination de l'effet d'un machine hydraulique inventée par M. Segner, professeur à Göttingue* (The determination of the effect of a hydraulic machine invented by Mr. Segner, a Professor at Gottingen), written in 1752 and published posthumously.

These works contain a detailed analysis of the functioning of this hydraulic machine. For Euler, working on these memoirs was also the occasion to apply in a particularly interesting practical instance his theoretical researches in mechanics, hydrodynamics, and the theory of motion. Let us say a few more words on them.

In the memoir [13], Euler discusses the ways in which the Segner machine can be used in a most efficient manner, showing the advantages that it is capable of providing, comparing it to other sources of energy based on water flow. He demonstrates that by employing the same quantity of water and the same fall, the new machine produces an effect about four times greater than the other known machines, when they are most advantageously used. Figure 1 is extracted from this memoir. In the memoir [15], Euler computes the optimal proportions of the Segner machine. In the posthumous memoir [16] he declares that the principles of this machine are completely new because the pipes through which the water passes are not at rest, but are subject to a motion caused by the force of the water itself. Thus, the usual laws of hydraulics are not sufficient for the study of this machine, and a completely new approach is needed. In the same memoir he gives an outline of the construction of a new water turbine which became the basis of his later study of a centrifugal pump.

Most of all, this collection of memoirs constitute a detailed mathematical study of a mechanical device based on differential calculus. In this study, the shape of the vase, the motion, the curvature and the other geometrical properties of the pipes play central roles. Euler states the problems that he solved in the form of optimization problems. For instance, Problem 6 asks the following:

To find the most advantageous figure that one can give to the horizontal pipes, for which the machine has the largest effect.

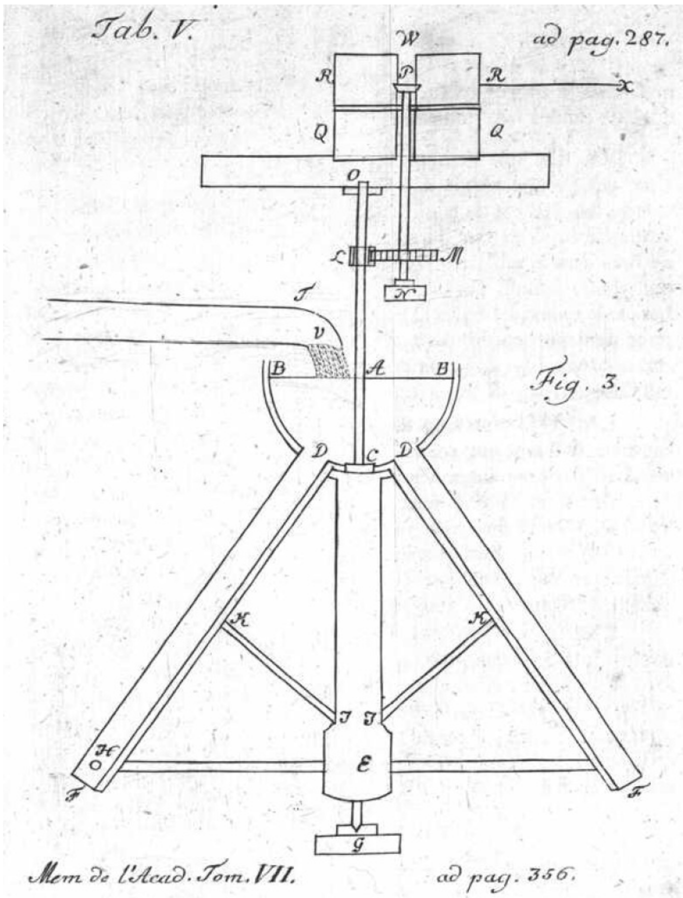

Figure 1. *From Euler's memoir on Segner's hydraulic machine [13]*

Among the other writings by Euler which concern machines and optimization, we mention his memoir *Maximes pour arranger le plus avantageusement les machines destinées à élever de l'eau par le moyen des pompes* (Maxims for the most advantageous arrangement of the machines designed to raise water by the use of pumps) [17]. Euler declares in this memoir that to set such a machine in motion, one can use the forces of water (rivers, etc.) or those of wind, or of men or of horses (he mentions incidentally that a single horse produces the same effect as a dozen men). His aim in his memoir is to find the most advantageous design for such a machine. He studies the various parameters that are involved with the aim of optimizing the results. For example, he searches for the speed with which the force, when it acts on the pumps that have to be put in action, produces the greatest benefit. The amount of water that a machine must raise also depends on the geometrical properties of the pipes. The shape and dimensions of these pipes are among the parameters that has to be optimized. In the same memoir, Euler considers other machines that are used to lift water and he presents applications to the functioning of windmills.

The Segner turbine, which we mentioned above, was considered as an important step in the later development of steam engines, see [18, p. 41]. Now we turn to the latter, with which Chebyshev was intensely concerned.

Chebyshev, like his famous predecessor, was most interested in mechanical engines. We learn from the biographies contained in his Collected Works [2] and from the article in his article *P. Tchébychef and his scientific work* by Vassiliev [19] that since his childhood, one of his favorite occupations was the construction of mechanical toys, and that he kept this passion during the rest of his life. Wooden automata, speed regulators, calculating machines, mechanical linkages and other devices he constructed at various stages of his life are displayed in a number of museums and science galleries, including the Conservatoire National des Arts et Métiers in Paris, the Academy of Sciences in Saint Petersburg, the Museum of History in Moscow, and the museum dedicated to Chebyshev, hosted in the school named after him in the village of Mashkovo, a few kilometers from his birth place.

As a student the University of Moscow, Chebyshev was highly attracted by the course on mechanics taught by N. D. Brashman (1796–1866), for whom he kept a profound respect, both as a mathematician and as a person. It is known that Chebyshev asked Brashman for a photograph that he always carried with him and that he still had it at the moment of his death. We also learn from Chebyshev's biography in [2] that after his arrival to Saint Petersburg, Chebyshev had close relations with another eminent mathematician, O. I. Somov (1815–1876), who was a specialist of applications of analysis and geometry to kinematics and the author of a geometric approach to theoretical mechanics. Let us also note that in addition to his work at Saint Petersburg's University and Academy of Sciences, Chebyshev taught applied mechanics at the Imperial Alexander Lyceum of Saint Petersburg.

Besides his interest in theoretical mechanics, Chebyshev, like Euler, was deeply interested in industrial machines. In a document reproduced in his Collected Works [2, vol. I, p. VII–XVIII], he reports on a trip he made to Western Europe in the summer of 1852, during which he spent a significant part of his time observing systems for raising water, agricultural machines, wind-mills, factories, cotton and flax spinning machines, metallurgical plants, foundries. He also visited several science museums in which he could contemplate steam engines. In the same report, Chebyshev makes detailed comments on the design of the wings of the wind-mills he saw during his journeys, expressing the desire to write analytical expressions for the surfaces of these wings in order to find the most economical way of using them in making oil, grinding grain, pounding flax, etc. We also learn from the same report that Chebyshev was particularly interested in the automata built by the French artist and mechanical engineer Jacques de Vaucanson (1709–1782), including automatic figures of the size of a human being that play music, models of digesting ducks which are capable of eating grains, then metabolizing and defecating them. Chebyshev was mostly fascinated by the steam engine invented by James Watt (1736–1819), the Scottish engineer who played a major role in the nineteenth century industrial revolution. He was particularly attracted by a piece of mechanism included in Watt's engine consisting of a mechanical linkage that transforms an almost circular motion of a rocker beam in the pumping part of the machine into the straight-line motion of a piston shaft. This linkage, to whom Chebyshev gave the name "Watt parallelogram", played a major role in his later work. It acted as a motivation for his investigations on approximation theory and on what became later known as the theory of linkages. In his report, Chebyshev makes a rather detailed review of the history of this linkage and its use in steam engines, explaining the reasons that led Watt to invent it and declaring that it is necessary to submit the Watt parallelogram and its variants to a rigorous scrutiny. Chebyshev declares that it is for this reason that he started studying conditions under which several elements of this mechanism in factory machines and on steamboats depend, and that, while he wanted to deduce the rules for the construction of these parallelograms, he encountered difficult problems of analysis which were still poorly studied. He says that the only person who tackled these problems was J.-V. Poncelet, the well-known French mathematician, engineer and general, and member the Academy of Sciences of Paris. Poncelet was known for his work in applied mechanics. Chebyshev points out that the theory of the Watt parallelogram, like several other problems in mechanics, requires formulae that are more general than those currently known.

As a matter of fact, Chebyshev's first published results on approximation theory are contained in his article on mechanical linkages titled *Théorie des mécanismes connus sous le nom de parallélogrammes* (The theory of mechanisms known under the name of parallelograms) [2, Vol. I, p. 111–143], published in 1854. In the introduction of this memoir, Chebyshev mentions, as a motivation for his work, the design of steam machines, and he explains that his aim in writing this article is to present a slight modification of Watt's parallelogram which reduces the deviation of the piston rod with respect to the pendulum of the machine. From the mathematical point of view, the problem needed approximation techniques which were much more elaborate than the existing ones. This led him to the basic question of finding a polynomial which approximates a function whose difference with the polynomial on a certain fixed small interval of the real line is minimal. Almost all the techniques that were available from analysis relied on the polynomials given by Taylor expansions, which give only approximations in the neighborhood of specific points, but they were not useful in the approximation problems along a whole interval. Legendre polynomials, introduced in 1782 by A.-M. Legendre, were already known, and they found applications in physics, but they had not been used in such approximation problems. This paper by Chebyshev contains his first investigations on what became later known as the Chebyshev polynomials, and the beginning of his theory of orthogonal polynomials. The basic question addressed in this paper, viz., that of finding a mechanical linkage that transforms a rectilinear motion into a circular one, gave rise to a large amount of works, by Chebyshev and others. This problem was eventually solved and by A. Peaucellier [20] and, later, by Chebyshev's student L. Lipkin [21]. One talks now about the Lipkin–Peaucellier inverter, a linkage having seven elements; see the report in [22].

Chebyshev wrote several other articles on the same subject, and we mention the following memoirs:

- *Sur une modification du parallélogramme articulé de Watt* (On the modification of Watt's articulated parallelogram) [2, Vol. I, p. 533–538] (1861);
- *Sur les parallélogrammes* (On parallelograms), [2, Vol. II, p. 85–106] (1870);
- *Sur les engrenages* (On gears) [2, Vol. II, p. 129–161] (1875);
- *Sur les parallélogrammes composés de trois éléments et symétriques par rapport à un axe* (On the parallelograms composed of three elements symmetrical with respect to an axis) [2, Vol. II, p. 285–297] (1879);
- *Sur les parallélogrammes composés de trois éléments quelconques* (On the parallelograms composed of three arbitrary elements) [2, Vol. II, p. 301–331] (1890);
- *Sur les plus simples parallélogrammes qui fournissent un mouvement rectiligne aux termes du quatrième ordre près* (On the simplest parallelograms that provide a rectilinear motion up to terms of degree four) [2, Vol. II, p. 359–374] (1882);
- *Sur la transformation du mouvement rotatoire en mouvement sur certaines lignes, à l'aide de systèmes articulés* (On the transformation of the circular motion into a motion on certain lines, using articulated systems) [2, Vol. II, p. 726–732], (1884); *Sur les parallélogrammes les plus simples symétriques autour d'un axe* (On the simplest symmetrical parallelograms that are symmetrical around an axis) [2, Vol. II, p. 709–714] (1885).

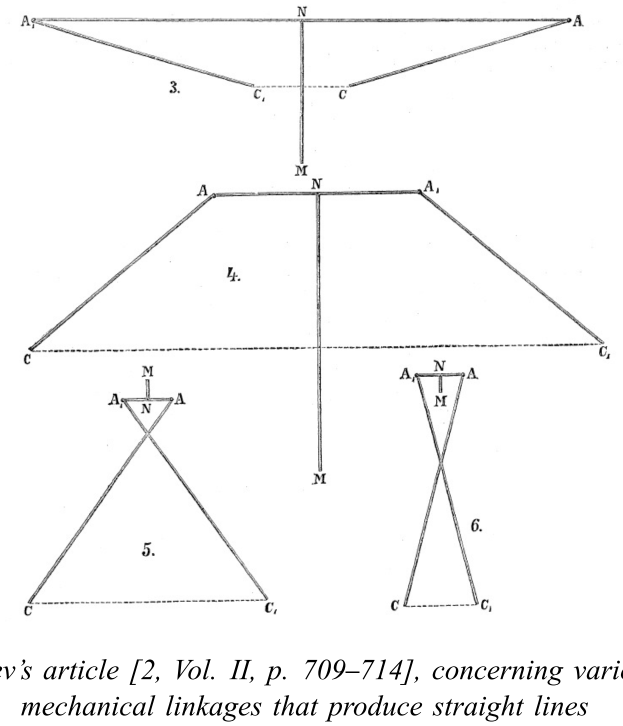

Figure 2. *From Chebyshev's article [2, Vol. II, p. 709–714], concerning various forms of symmetrical mechanical linkages that produce straight lines*

Chebyshev presented his results on linkages at several meetings of the Saint Petersburg Academy of Sciences, but also at the Association française pour l'avancement des sciences, a learned society whose aim was to promote science at various levels, connecting scientists and science amateurs coming from different backgrounds and establishing links between various fields of knowledge. Chebyshev participated to several meetings of this society, which were held in the summer, each year in a different city in France. He also presented his mechanical linkages and other machines he invented at the Columbian World's Fair (Chicago, 1893) and at the *Musée de l'École des Arts et Métiers* in Paris.

Like Euler, Chebyshev approached systematically the theory of machines using mathematical methods. Conversely, his study of linkages acted as a stimulus for his mathematical work on the theories of approximation and optimization.

Vassiliev in [19] reports that Chebyshev, for his work on the approximations of functions was not only motivated by problems raised in the theory of mechanisms, but also by problems on ballistics. In the next section, we shall report on Chebyshev's work on this topic, but before, that, we shall review Euler's work on the same subject.

Ballistics

I will start by mentioning a book by the English mathematician and military engineer Benjamin Robins (1707–1751). The latter was a student of Henry Pemberton, who was Newton's disciple and the editor of the third edition of the *Principia* (1726). Robins is well known for his invention of a method for measuring the speed and the kinetic energy of a projectile from the effect of its impact on a pendulum which was given the name ballistic pendulum. He was familiar with Euler's works on the theory of motion, and in 1739 he published an article titled *Remarks on Mr. Euler's Treatise of motion, Dr. Smith's complete system of optics, and Dr. Jurins essay upon distinct and indistinct vision* in which he discussed this theory, see [23]. Robins used several ideas of Euler on motion in his later works.

In 1742, Robins published a book titled *New Principles of Gunnery* [24]. Euler was very much interested in this topic, in which several of his ideas in physics could find their application. Indeed, gunnery involves the theory of motion, the nature of air and fire, the physics of air resistance, elasticity, fluid mechanics, the study of the dependence of air resistance and elasticity on temperature and their impact on the velocity of a projectile, etc. Furthermore, this was a time where Euler was working at the Berlin Prussian Academy under the auspices of Frederick the Great, and gunnery was a subject which was of high importance for the Prussian ruler.

Euler translated Robin's book into German, and he presented it, together with his own commentary, to the Berlin Academy of Sciences, in 1753, under the title *Neue Grundsätze der Artillerie* (New Principles of Gunnery) [25]. This translation, with the commentary, were in turn translated into English again, and published in 1777, by the British ballistician Hugh Brown, see [26]. A French edition of the same book, together with Euler's comments, was published in 1783 under the title *Nouveaux principes d'artillerie de M. Benjamin Robins, commentés par M. Leonard Euler* (New principles of artillery by Mr. Benjamin Robins, commented by Mr. Leonard Euler) [27].

In Euler's edition of Robin's work, each proposition of the latter is followed by several complements by Euler. These complements sometimes consist in small new sections proving the same results, in which Robins' geometrical methods are replaced by analytical methods, using the tools of calculus and algebra, which Euler considered easier for the student. Euler's complements also involve the introduction of tools from infinitesimal calculus and ideas originating in physics, from the study of air resistance and its elastic properties to that of the force of powder explosions. We note that Euler had already published several memoirs related to this topic, and we shall mention some of them below.

Let us give now a few examples of Robin's results.

Proposition VII of the first Chapter deals with the following problem:

> *Given the length and the diameter of a piece of cannon, together with the weight of the bullet, the charge of the powder and its elastic force at the first moment of the explosion, to determine the velocity with which the bullet is projected out of the cannon.*

Proposition VI of Chapter 2 says the following:

> *The curve described in the air by a bomb, or a bullet, is not a parabola, and it is even not close to a parabola, unless the velocity of the body projected is very small.*

In fact, Robins did not give the nature of that curve. Euler, in three different addenda to this proposition, developed the same subject based on his own researches on air resistance, treating three cases separately: the motion of a body projected horizontally, then vertically, and finally obliquely. In each case he determined and studied the nature of the curve obtained.

In 1727, Euler, who was only 21 years old, wrote a paper titled *Meditatio in experimenta explosione tormentorum nuper instituta* (Meditation on experiments made recently on the firing of a cannon) [28]. In this paper, he describes seven experiments on gunnery performed between August 21 and September 2, 1727. The paper is known also because it is the first time that the letter *e* is used to represent the basis of natural logarithms. It was published posthumously, only in 1862 (that is, 79 years after Euler's death).

In 1755, Euler published a memoir on ballistics titled *Recherches sur la véritable courbe que décrivent les corps jetés dans l'air ou dans un autre fluide quelconque* (Researches on the real curve described by the bodies thrown into the air or into an arbitrary fluid) [29]. In this memoir, he starts by recalling that Galileo Galilei's discovery of the fact that a body projected obliquely in the air describes a parabola cannot be used in artillery, because the velocity of the projectiles that are used in this field (i.e. bombs, bullets, etc.) is very high, and air resistance is too large compared to gravity. Precisely, velocity and air resistance are the two parameters that divert projected bodies from following a parabolic trajectory, as they would do in vacuum. In this paper, Euler studies in detail these questions using the tools he had already developed in mechanics, improving results that were obtained by Johann Bernoulli. The latter had developed the mathematical aspect of the question of the trajectory of a bullet submitted to air resistance under the assumption that the bullet is subject to a force which is proportional to the square of its velocity, in addition to the gravitational force.

The French version of Euler's edition of Robins' book contains, as an appendix, excerpts from a memoir of Euler published in 1729, titled *Tentamen explicationis phaenomenorum aeris* (An essay on the explanation of the phenomena of air) [30], in which the author elaborates on his theory of the atmosphere.

In the nineteenth century, the field of ballistics was as much important as it was in the eighteenth. Vassiliev, in his biography [19, p. 10], writes that when Chebyshev was appointed on the chair of applied mathematics at the Saint Petersburg Academy, he was concerned with two problems in applied mathematics: the theory of mechanisms, and ballistics.[1] In fact, these are the two subjects that led Chebyshev to the development of the theory of approximations. From the same article, we learn that it was a problem in ballistics that was proposed to Chebyshev by his hierarchy in the artillery administration, that led him to concentrate on questions of interpolation [19, p. 17]. His written work on ballistics is reproduced in Volume V of the Russian edition of his complete works. An English summary is given in the article [31]. We can read there that in 1856 Chebyshev became a member of the Artillery committee of the Imperial Russian army, a committee which was in charge of introducing scientific novelties in Russian artillery. Three years after his nomination at that committee, Chebyshev was awarded the second-degree of the St. Anne Order, a highly valued medal, for his "three-year long meticulous and useful work, in addition to performing his direct duties at the Academy of Sciences, on mathematical research at the Artillery Committee, a work which turned out to be so beneficial for the development of rifle guns and shells." In 1867, Chebyshev obtained a formula for the range of spherical shells whose initial velocities are within a certain limit. His works on interpolation theory was in part motivated by the computation of a table on the effect of bullets based on experimental data. His Russian collected works edition [3, Vol. 5, p. 239] contain a document extracted from the Saint Petersburg Military Scientific Council of 1863, displayed in the Artillery museum of Saint Petersburg, presenting the difficult problems of artillery that needed to be solved using mathematical tools. We read there:

> The efforts of officers well-versed in general mathematical sciences are not enough and call the assistance of great geometers. For many centuries outstanding geometers of all countries have been engaged in artillery studies and it is to them alone, of course, that artillery as a science owes the precious stock of rational and reasonable ideas.

The document's conclusion reads:

> In the future, the Artillery Committee will be still very much in need of Chebyshev's assistance; if he abandons artillery, the committee will be compelled either to abandon the investigation into mathematics-based issues or turn to the mercy of Mr. Chebyshev, who, with no direct obligations before the committee, will have no reason to devote his precious hours of scientific research to artillery questions.

Chebyshev's results also include the application of probability theory in artillery, an explanation of the role of rotary motion in shell fire and the study of the optimal shape of a shell.

[1] As a matter of fact, Chebyshev was elected in 1853 on the chair of applied mathematics at the Saint Petersburg Academy, because the chairs of pure mathematics were already full (they were occupied by Fuss, Ostrogradsky, and Bunyakovsky). This reminds us of the fact that Euler, at his arrival in St Petersburg in 1727, was offered the chair of physiology, because the chair of mathematics-physics was occupied (by Daniel Bernoulli).

I will end this section with two pictures on ballistics. The first one, Figure 3, reproduces a drawing by Leonardo da Vinci extracted from the Codex Atlanticus.

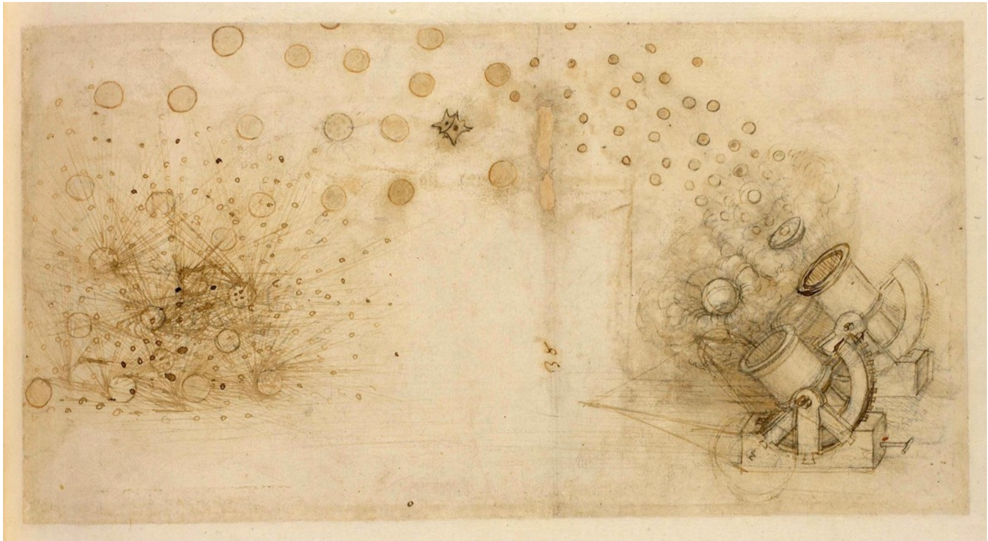

Figure 3. *From Leonardo's Codex Atlanticus*

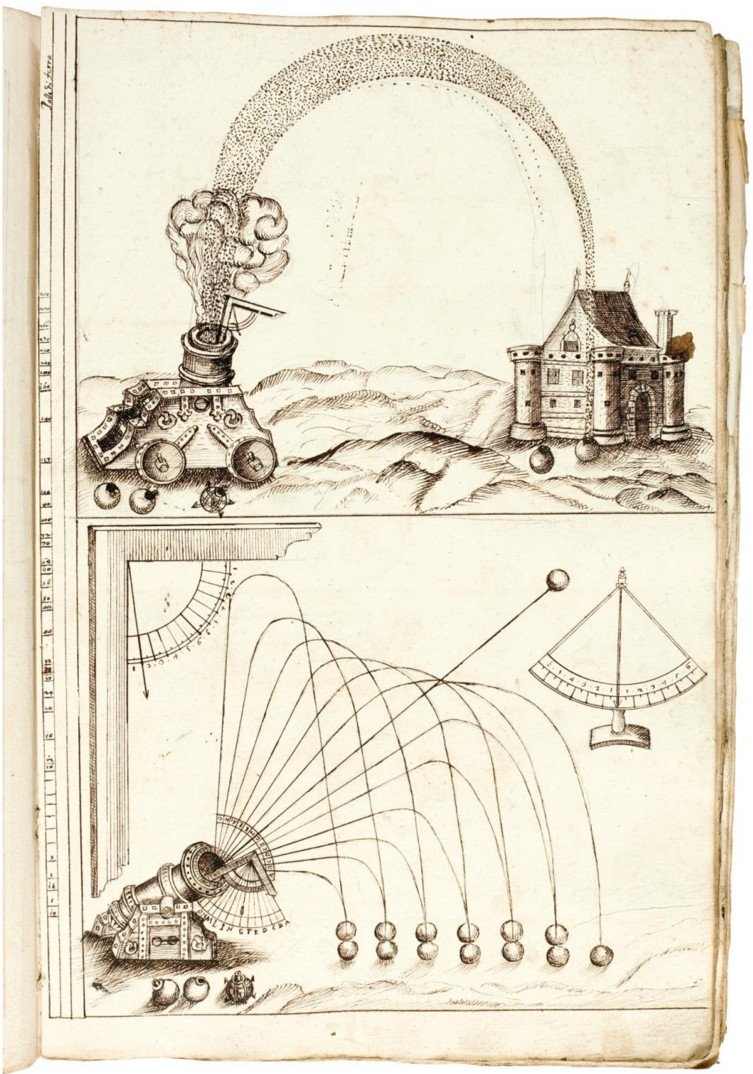

Figure 4. *From a 17th century manuscript on geometry and artillery.*

The second one, Figure 4, is also an Italian drawing. It is reproduced from an 17th century manuscript on geometry and artillery titled *Di alcuni principi di Geometria necessarij per intendere, e sapere fare spedissamente tutto quello che si appartiene ad un valoroso Bombardiere* (Some principles of Geometry necessary to understand and know how to do very quickly everything that belongs to a valuable bomber). The drawing is inspired from a drawing by the Spanish military writer and artist Diego Ufano.

We now pass to geography.

Geography

In Euler's times, the geography department at the Saint Petersburg Academy of Sciences was one of the most important departments. This section (like the one of astronomy) belonged to the class of mathematics, and it was headed by the famous French geographer Joseph-Nicolas Delisle. One of his main tasks was to make precise measurements of the Russian Empire and to draw geographical maps. For this, Delisle needed the help of mathematicians. He was first assisted by Daniel Bernoulli and Jakob Hermann, but later, these two left Russia and Euler became his main collaborator. Eventually, the relations between Delisle and the Academy's administration became tense, and Euler became himself the leader of the geography department. As such, he became responsible for the publication of the so-called *Atlas Russicus* (the "Russian Atlas"), a project which was one of the major enterprises at the Academy and of which Delisle was in charge previously. The *Russian atlas* was eventually published under the direction of Euler, in 1745. It carried the title *Atlas Russicus mappa una generali et undeviginti specialibus vastissimum Imperium Russicum adjacentibus regionibus secundum leges geographicas, et recentissimas observationes delineatum exhibens* (Russian Atlas containing a general map and nineteen particular maps of the whole Russian Empire and its limiting countries in conformity with the rules of geography and the most recent observations), see [32]. It contains 20 maps, including a general map of the Russian empire and nineteen particular maps.

In Figures 5 and 6, we have represented two of these maps.

Figure 5. *Map of the Russian Empire, extracted from Euler's Atlas Russicus Saint Petersburg, 1745 [32], Bibliothèque nationale de France, Département Cartes et plans*

In 1753, Euler published another atlas, this time under the auspices of the Prussian Academy of Sciences in Berlin where he had settled [33]. The atlas was meant essentially for the use of school children. It contains 44 maps, representing all the regions of the Earth. One should remember in this respect that a fierce international competition was taking place in that period, for the drawing and publishing of maps of Russia and its bordering countries (in particular China and Japan). Furthermore, in both academies, the selling of individual maps was a non-negligible source of income. In Figure 7, we have reproduced a map from the Prussian atlas, representing the land of Palestine. This map was especially intended for religious studies classes.

Source gallica.bnf.fr / Bibliothèque nationale de France

Figure 6. *Map of part of Siberia, extracted from Euler's Atlas Russicus, Saint Petersburg, 1745 [32], Bibliothèque nationale de France, département Cartes et plans*

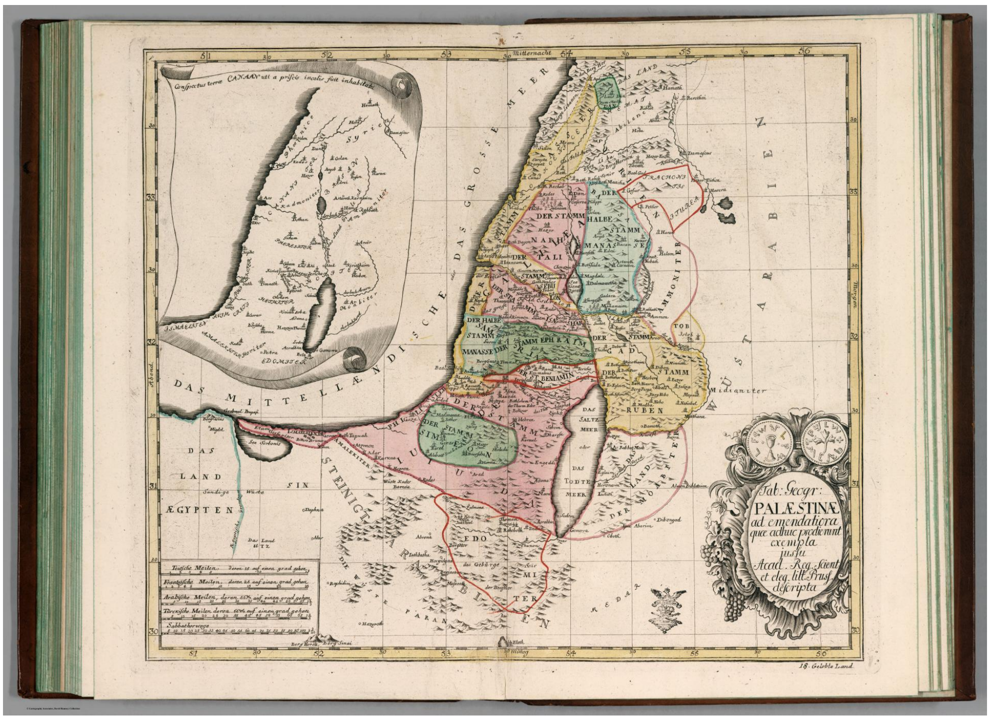

Figure 7. *Map of part Palestine, extracted from Euler's Atlas Geographicus, Berlin 1753 Atlas, David Rumsey Map Collection at Stanford University Libraries*

Euler wrote several articles related to geography. We briefly review some of them, explaining the context.

In the last quarter of the seventeenth century, the sphericity of the Earth was challenged by Newton, who deduced from his gravitation theory developed in the *Principia*, that the Earth was rather spheroidal than spherical. A spheroid is a figure obtained by the revolution of an ellipse along one of its two axes. More precisely, Newton considered that the Earth is a spheroid whose axis is the axis around which the Earth actually revolves, and that its form is that of an orange, flattened at the top and at the bottom. A large debate around this question kept busy the scientific community of the Eighteenth century during several decades, a debate in which Euler was involved. A conflict arose between, on the one side, supporters of the Newtonian theory, including Delisle and Euler, and on the other hand, the “French geographers” led by the influential astronomers of the Cassini family, in charge of the Paris Observatory, who considered on the contrary that the Earth has the form of a spheroid which, like an egg, is elongated at the poles and flattened at the equator.

With the appearance of the new theories of the figure of the Earth, many geographical and astronomical results that were obtained during the preceding centuries needed to be revised. Euler wrote several geometrical memoirs on this subject, some of them theoretical, and others involving practical methods: finding the geodesics and measuring distances on a spheroid, drawing maps of regions on the spheroidal Earth, etc. Even elementary notions needed to be modified. For instance, with the Earth being considered as spheroidal and not spherical, the length of an arc of a meridian depends on the latitude.

In the memoir *Methodus viri celeberrimi Leonhardi Euleri determinandi gradus meridiani pariter ac paralleli telluris, secundum mensuram a celeb. de Maupertuis cum sociis institutam* (Method of the celebrated Leonhard Euler for the determination of a degree of a meridian, as well as of a parallel of the Earth, based on the measurement undertaken by the celebrated de Maupertuis and his colleagues) [34] (1750), Euler studies several problems in this new setting, including the determination of the length of a degree of a meridian from the latitude at a point.

In the memoir *De attractione corporum sphaeroidico-ellipticorum* (On the attraction of spherico-elliptical bodies) [35] (1747), he obtains a formula for the attraction between a particle situated at a pole and another one situated at the equator, taking into account the spheroidal shape of the Earth. His memoir *Élémens de la trigonométrie sphéroïdique tirés de la méthode des plus grands et plus petits* (Elements of spheroidal trigonometry drawn from the method of the maxima and minima) [36] contains the bases of spheroidal geometry. In particular, Euler gives there the formula for the distance between two locations on the surface of the Earth, given their latitude and longitude. Of course, the solution of this problem is much more involved than the analogous one in the case of a spherical Earth. One complication arises from the fact that the perpendicular line to the tangent space at a point of a spheroid does not pass through the center, except if this point is on the equator or at the poles, therefore the latitude at a point is equal to the angle that this perpendicular makes with the rotation axis of the Earth.

In the memoir titled *On the parallax of the moon with respect to its elevation and azimuth, under the hypothesis of a spheroidal earth* [37] (1751), Euler studies an astronomical problem in the setting of the spheroidal shape of the Earth. The term “parallax” in the title refers to the influence of the position of an observer on an object he sees in the sky. The object is, here, the moon. From the mathematical viewpoint, this is a coordinate change problem, in spheroidal geometry.

In 1738, Euler published an exposition, intended for general audience, of the problems represented by the shape of the Earth, *Von der Gestalt der Erden* (On the shape of the Earth) [38].

In 1777, he published three memoirs on the mathematical problem of drawing geographical maps, *De repraesentatione superficiei sphaericae super plano* (On the representation of spherical surfaces on the plane) [39], *De proiectione geographica superficiei sphaericae* (On a geographic projection of the surface of the sphere) [40] and *De proiectione geographica De Lisliana in mappa generali imperii russici usitata* (On Delisle’s geographic projection and its use in the general map of the Russian Empire) [41]. These memoirs constitute a major advance in cartography. Besides giving a firm basis to Delisle’s methods of map drawing, Euler systematically introduced in this field the use of differential equations and the methods of the calculus of variations. Together with Lagrange, who also published two long memoirs on the construction of geographical maps [42], Euler included this subject in the general setting of mappings between differentiable surfaces. The forthcoming book [43] contains a detailed account of Euler’s works on geography and cartography, with their relation with other important eighteenth century works on the same subject.

Let us note that besides his work on the theoretical problems of cartography, Euler's duties at the Saint Petersburg and the Berlin Academies included the organization of the sale of geographical maps for the benefit of these academies. He was also asked to examine and to check thousands of maps that were printed there. This is mentioned at several places in his correspondence with the administration of the Academies, where we can also read that Euler became very annoyed with such tasks, and that he considered that the examination of so many maps was the reason for the deterioration of his vision. In a letter to Christian Goldbach, dated August 21st (September 1st, new Style), 1740, he writes (cf. [44] p. 163, English translation p. 672):

> Geography is fatal to me. As you know, Sir, I have lost an eye working on it; and just now I nearly risked the same thing again. This morning I was sent a lot of maps to examine, and at once I felt the repeated attacks. For as this work constrains one to survey a large area at the same time, it affects the eyesight much more violently than simple reading or writing. I therefore most humbly request you, Sir, to be so good as to persuade the President by a forceful intervention that I should be graciously exempted from this chore, which not only keeps me from my ordinary tasks, but also may easily disable me once and for all.

From the purely geometrical point of view, the problem of drawing geographical maps is that of studying maps from a curved surface (in general, the sphere or a spheroid) onto a flat surface (the Euclidean pane). This gives rise to several interesting geometrical questions. Chebyshev, almost a hundred years later, became most interested in this subject, and we shall talk now about his work.

In 1856, Chebyshev wrote two memoirs on cartography with the title *Sur la construction des cartes géographiques* (On the construction of geographical maps) [2, Vol. 1, p. 233–236 and 239–247]. The title is the same as that of the two memoirs that Lagrange published fifty-seven years earlier [42]. Chebyshev's main technical notion that he used in this work is a mathematical concept of infinitesimal dilatation of a conformal map, which is based on a formula that Lagrange obtained. He introduced a notion of distortion of the map, which is a measure of how much the infinitesimal dilatation differs from its mean values over all the region represented. Chebyshev addressed the question of finding geographical maps whose distortion is minimal. Whereas Lagrange worked in the setting of a spheroidal Earth, Chebyshev considered the simpler case where the Earth is spherical. This mathematical problem of finding maps from the sphere to the Euclidean plane with minimal distortion is typically a problem in the calculus of variations, a field which was founded by Euler and Lagrange. Chebyshev introduced a new ingredient in this study, namely, the Laplace equation, establishing relations between the construction of geographical maps and the question of heat equilibrium. His main result in this domain reduced the search for the best geographical map to that of finding a solution of the Laplace equation with given boundary conditions. More precisely, he proved that for any a region of the sphere (representing a country), this distortion is smallest when dilatation on the boundary of the region is constant [2, t. I, p. 242].

In his memoirs on geography, Chebyshev compares his result in this domain with those that he obtained on linkages. He writes [2, p. 242]:

> This problem is similar to those which were the subject of my dissertation on Watt's parallelogram, but it is related with a higher order such problems: one was looking for certain constant quantities, whereas here one must find two unknown functions, which corresponds to the determination of an infinite number of constant quantities. This establishes a distinction between these problems similar to the one that exists between the problems of differential calculus and those of calculus of variations. This subject is all the more interesting from a theoretical point of view because it is related to the investigation of an extremely remarkable partial differential equation, which expresses, for example, also the heat equilibrium in infinitely thin membranes. Thus the question of the best projection is related to the remarkable property of heat, namely: when heat is in equilibrium in an infinitely thin circular membrane, the temperature at the center is an *average* of the temperatures at all points on the circumference; a similar proposition is true for the sphere: the temperature at the center is an *average* of the temperatures at the surface.

We now point out another problem on geography on which Chebyshev worked, and which is analogous to a problem considered by Euler. We recall that the latter, in several memoirs, gave formulae, based on trigonometric computations, giving the distances between two points on the Earth if one knows their latitude and longitude. In a note titled *Chebyshev's rule for the approximate evaluation of distances on the surface of the Earth*, published in the second volume of his collected works, Euler gave a practical rule for giving an approximation of this distance, avoiding the long trigonometric formulae. He first addressed the question: [2, t. II, p. 736]:

> *Given two locations on the surface of the Earth (considered to be spherical), determined by their geographical coordinates, that is, their longitude and latitude, to compute their actual distance.*

He then gives the rule:
1) Express in minutes the difference in longitudes and latitudes of the two locations;
2) Take the double of the difference in latitudes;
3) From the two numbers, namely, the difference of longitudes and the double of the difference in latitudes, multiply the smallest by 3 and the largest by 7, and add the result.
4) The sum of the resulting two numbers, divided by 8, will give the required distance (in versts).[2]

I also discuss Chebyshev's work on cartography, making relations with works of Euler and with other topics, in the papers [1, 45–49] and the chapter of the forthcoming book that will be published on the occasion of the 2022 ICM in Saint Petersburg, [50].

I will pass now to works of Euler and Chebyshev on pure mathematics.

Integrals

Generally speaking, in the works of Euler and Chebyshev which we shall discuss in this section, integration was considered as the operation inverse to differentiation. We naturally start with Euler. The study of integrals in his work was intimately related to that of differential equations. Finding a formula for an indefinite integral $\int f$ is equivalent to solving a differential equation of the type $dy/dx = f(x)$. Of course, the modern idea of measure theory was not known, even though the geometric interpretation of the integral as the area of a region below a curve, or as the length of a curve, was known to Euler's; we shall talk about this soon. In fact, this subject has a long history in which the Bernoullis, Huygens and even earlier mathematicians were involved.

Euler wrote more than a hundred memoirs on integration. The three-volume treatise on the foundations of integral calculus, *Institutionum calculi integralis* (Institution of integral calculus) [51–53] contains many of his results on indefinite integrals and on differential equations. Among the huge class of integrals of functions that he studied, elliptic integrals play a major role. Between 1738 and 1882, he wrote more than 30 memoirs on these integrals. We shall say a few words about Euler's work on such integrals (together with their generalizations to the so-called Abelian integrals). Chebyshev also worked on this subject.

We start with an analogy. Whereas the integral $\int \frac{dx}{\sqrt{1-x^2}}$ represents the length of an arc of a circle centered at the origin, the integral $\frac{dx}{\sqrt{1-x^4}}$ represents the length of an arc of another curve in the plane (a lemniscate of polar equation $r^2 = \cos 2\theta$). More generally, the theory of elliptic integrals involves the study of properties of some integrals of plane curves (the term elliptic refers to the special case of an ellipse), and in this topic, one looks for properties that are analogous to those of integrals representing arcs of circles, that, in fact, are the trigonometric functions. For instance, one searches for sums and product formulae, for the existence of periods, etc. The subject of elliptic integrals can be traced back to Giulio Carlo de' Toschi di Fagnano and to Johann I Bernoulli. The latter was Euler's teacher in Basel. Finding general addition theorems for elliptic integrals became one of the major problems for the following hundred years, with works of Euler, Legendre, Abel, Jacobi and Riemann. Legendre, in his *Traité des fonctions elliptiques et des intégrales eulériennes* (1825–1828 3 volumes), highlights a special class of such integrals he calls Eulerian.

Let us mention a few memoirs by Euler on integration, whose title is always informative.

- *Specimen de constructione aequationum differentialium sine indeterminatarum separatione* (Example of the construction of differential equations without separation of the indeterminates) (1738) [54], where Euler gives a formula for arc lengths of ellipses;

[2] The verst is an ancient measure of distances used in Russia, equal to 1066.8 meters.

- *De integratione aequationis differentialis* $mdx/\sqrt{1-x^4} = ndy/\sqrt{1-y^4}$ (On the integration of the differential equations $mdx/\sqrt{1-x^4} = ndy/\sqrt{1-y^4}$) (1761) [55];
- *Observationes circa integralia formularum* $\int x^{p-1}dx(1-x^n)^{q/n-1}$ *posito post integrationem* $x = 1$ (Observations concerning the integrals of formulae$\int x^{p-1}dx(1-x^n)^{q/n-1}$ setting $x = 1$ after integration) (1766) [56];
- *Speculationes super formula integrali* $\int \frac{x^n dx}{a^2-2bx-cx^2}$*, ubi simul egregiae observationes circa fractiones continuas occurrunt* (Speculations concerning the integral formula $\int \frac{x^n dx}{a^2-2bx-cx^2}$, where at once occur exceptional observations about continued fractions) (published in 1786) [57]; in this memoir Euler uses continued fractions in problems of integration. We shall consider continued fractions in the next section.

In the nineteenth century, the study of indefinite integrals and the question of finding explicit formulae for integrals of some special functions in the huge collection of arbitrary functions was a fashionable topic in Russia (as it was in France). We recall incidentally that Lobachevsky, in struggling for the recognition of his discovery of non-Euclidean geometry, endeavored to find a large number of indefinite integrals, each time by calculating areas and volumes of certain domains of the hyperbolic plane or hyperbolic 3-space in two different manners and putting the results to be equal. Some of his papers contain a large number of such integrals. For him, this was a matter of finding applications of hyperbolic geometry.

We pass now to Chebyshev.

One of the very first works of Chebyshev is on integration theory. In 1843, he published in Liouville's journal a paper on multiple integrals titled *Note on a class of multiple indefinite integrals* [2, p. 3–6]. In this paper, he obtained new proofs of formulae of Liouville and Cauchy on the multiple integration of some special functions. The work immediately attracted the attention of French mathematicians. In the same issue of the journal, Catalan published a 2-page complement to Chebyshev's result, of which, therefore, he was aware, before the paper appeared in print.

In 1847, Chebyshev moved from Moscow to Saint Petersburg where he submitted a *pro venia legendi* dissertation titled *On integration by means of logarithms*. The problem he addressed was to express some indefinite integrals of the form

$$\int \frac{x + Adx}{\sqrt{x^4 + ax^3 + bx^2 + cx + d}}$$

in terms of logarithms of algebraic functions. This problem was posed by M. V. Ostrogradsky, one of the prominent mathematicians in Saint Petersburg. Chebyshev was working on this question since 1843, that is, since the publication of the paper we mentioned above. In his 1852 report on his journey to Western Europe which we quoted in §, Chebyshev writes that he chose the subject of this dissertation on the instigation of Liouville and Hermite. He mentions that such integrals involving only square roots of rational functions are the simplest and at the same time, they are most frequent in applications. In the same dissertation, he presented the state of the art for the general problem of integrating irrational differentials, quoting works of Jacobi, Abel and others. The dissertation was published posthumously in the Russian edition of Chebyshev's *Collected Works* [3, Vol. V]. A summary in French is contained in the French edition [2, Vol. II, p. xix–xx].

Chebyshev published later on a long series of papers on indefinite integrals, several of them in Liouville's journal. Along the years, he studied integrals of more and more complicated functions. We mention the following papers:

- *Sur l'intégration des différentielles irrationnelles* (On the integration of irrational differentials) [2, Vol. I, p. 147–168] (1853), in which Chebyshev presents a method for finding an algebraic expression for integrals of differentials of the form

  $$R(x)\frac{dx}{\sqrt[m]{\theta x}}$$

  where $R(x)$ is a rational function of x (a quotient of two polynomials) and θ a polynomial. His goal is to see whether these integrals are expressible by simple functions, and to investigate the number of logarithmic parts they involve. He uses techniques introduced by Abel and Liouville, which give representations of such integrals as infinite series. His article contains proofs of statements that were made without proof by Abel.

- *Sur l'intégration des différentielles qui contiennent une racine carrée d'un polynôme du troisième ou du quatrième degré* (On the integration of differentials that contain a square root of a polynomial of the third or fourth degree) [2, p. 171–200] (1857) in which Chebyshev shows how to find expressions for the integral
$$\int \frac{x+L}{x^4+lx^3+mx^2+nx+p}dx$$
under some hypothesis on the decomposability of the polynomial in the denominator, using a method of Abel involving a development in continued fraction of this function and searching for periodicity in this development.
- *Sur l'intégration des différentielles irrationnelles* (On the integration of irrational functions) [2, Vol. I, p. 511–514] (1860), which is a continuation of the preceding article, where Abel's methods are also used;
- *Sur l'intégration de la différentielle* $\int \frac{x+A}{\sqrt{x^4+\alpha x^3+\beta x^2+\gamma x+\delta}}dx$ (On the integration of the differential $\int \frac{x+A}{\sqrt{x^4+\alpha x^3+\beta x^2+\gamma x+\delta}}dx$) [2, Vol. I, p. 517–530] (1861) in which Chebyshev gives a method in the case where the coefficients $\alpha,\beta,\gamma,\delta$ are rational;
- *Sur l'intégration des différentielles qui contiennent une racine cubique* (On the integration of differentials containing a cubic root) [2, p. 563–608] (1865);
- *Sur l'intégration des différentielles les plus simples parmi celles qui contiennent une racine cubique* (On the integration of the simplest differentials among those which contain a cubic root) [2, Vol. II, p. 43–47] (1867), in which Chebyshev shows that one cannot find explicit finite formulae for integrals of the form $\int \frac{\rho}{\sqrt[3]{x^3+ax+b}}dx$.

We now pass to continued fractions.

Continued fractions

We mentioned in the preceding section that Euler used the method of continued fractions in his work on integration which Chebyshev, later, used extensively. We make some comments on this method.

A continued fractions is originally an algorithm for finding an approximation of a real number by a sequence of rationals. This notion is contained in Euclid's *Elements* (see Propositions VII.1, VII.2 and X.2). It is the method that the Greeks (who called it *Anthyphairesis*) used to define incommensurability.

The modern theory of continued fractions starts in the work *L'algebra parte maggiore dell'aritmetica* (1572) by Rafael Bombelli, which this author uses to find approximations of square roots. A century later, John Wallis, in his *Arithmetica infinitorum* (1656) used continued in a more systematic way.

Euler started working on continued fractions in the early 1730s, but his first memoir on the subject, *De fractionibus continuis dissertatio* (A dissertation on continued fractions) [58], was published in 1744. In this memoir, he used continued fractions to solve the Riccati differential equation. At the same time, the memoir contains an introduction to the theory of continuous fractions, in which Euler also informs his reader that he was working on this subject since a long time, despite the fact that he had not published anything. Incidentally, he mentions the work of Wallis on this subject, and he also recalls that the method originates in the Euclidean algorithm. He notes that even though this notion is less used than infinite series or infinite products, it is a very efficient tool for obtaining approximations. After giving examples of continued fraction expansions (in particular for $\sqrt{2}$, $\sqrt[3]{2}$ and π), Euler gives a proof of the fact that periodic continued fractions arise from fractions that are solutions of quadratic equations. He then looks for patterns that are more general than periodicity, in particular, sequences displaying arithmetic progressions. He observes that the continued fraction expansion of e and its powers exhibit such special patterns, and he develops a method of finding the value of a continued fraction expansion whose denominators form an arithmetic progression. He then presents the solution of the Riccati equation in the form of a continued fraction expansion.

Euler's famous treatise *Introductio in analysin infinitorum* (Introduction to the analysis of the Infinite) [59, 60] (first volume published in 1748) contains several sections on continued fractions. In fact, the subject of this book consists of three major topics: infinite series, infinite products and continued fractions.

The memoir *De fractionibus continuis observationes* (Observations on continued fractions) [61] (1750) is a self-contained treatise on continued fractions and their applications, in which Euler claims that this is a completely new theory which is useful in all branches of analysis. There are many other memoirs of Euler on continued fractions, including several on the solution of the Pell equation, in which continued fractions are used in an essential way.

Like Euler, Chebyshev used continued fractions expansions of functions in his study of integrals of certain irrational functions. His paper titled *On continuous fractions* [2, Vol. I, p. 201–230] contains the foundations of his theory of orthogonal polynomials. In this paper, presented in 1855, at the Saint Petersburg Academy, the method of least squares, continued fractions and Chebyshev polynomials are used to obtain an interpolation formula which gives a polynomial function approximating a certain function on a certain function on an interval, from the values at certain points. The paper was translated into French by Bienaymé and published in Liouville's Journal in 1858. A few years later, Chebyshev's wrote to Brashman a letter carrying the title *On algebraic continued fractions*. The letter was read on September 30, 1865 at a meeting of the Moscow Mathematical Society. It was published in the same year in Liouville's Journal [2, Vol. 1, p. 611–614]. This letter is a sequel to the article *On the interpolation by the method of least squares*. In the introduction to the letter, Chebyshev reports on the use of algebraic continued fractions, together with the method of least squares, in interpolation theory. He shows that given two function u and v, one can find, by developing u in continued fractions, two polynomials X and Y such that the expression

$$uX - Y$$

is the closest possible to v. At the same time, he gives a method for finding the two polynomials X and Y. He further developed the same ideas in his memoir *Sur le développement des fonctions en série an moyen des fractions continues* (On the development of functions in series using continued fractions) [2, Vol. 1, p. 617–636] (1866), and in his much later memoir *Sur une série qui fournit les valeurs extrêmes des intégrales, lorsque la fonction sous le signe est décomposée en deux facteurs* (On a series that gives the extremal values of integrals, when the function is decomposed into two factors) [2, Vol. 2, p. 405–417] (1883).

Other interpolation techniques are developed in his paper *On the interpolation by the method of least squares* (1859) [2, Vol. 1, p. 473–498]. We finally mention the paper *On interpolation in the case of a great number of data obtained by observation* (1859) [2, Vol. 1, p. 387–469] whose title is self-explanatory.

Let us conclude this section by mentioning that Chebyshev also used continued fractions in his fundamental work on probability theory, see the paper *Sur l'interpolation des valeurs équidistantes* (On the interpolation of equidistant values) [2, Vol. 2, p. 219–242] (1875).

Number theory

I will end this paper by a few words on number theory, one of the major topics on which both Euler and Chebyshev worked. Of course, it is not possible to give an idea of the broadness of their works on this subject, and I will limit myself to a few remarks, indicating how Chebyshev was the heir of Euler in this field.

In fact, Chebyshev, very early in his career, examined all Euler's work on number theory. Indeed, at the time he became adjunct professor at the University of Saint Petersburg, the Academy of Sciences had started the far-reaching project of publishing Euler's collected works on number theory. The project was led by Bunyakovsky who, when he noticed Chebyshev's talents, proposed him to contribute to that edition. The 1849, two large volumes were published under the direction of Bunyakovsky and Chebyshev, carrying the title *Index systématique et raisonné des oeuvres arithmétiques d'Euler* (Systematic and rational index of the arithmetical works of Euler). The volumes included Euler's published paper on the subject (99 memoirs) together with annotations and corrections. (Indeed, several mistakes had been introduced in Euler's paper that were published in the last 17 years of Euler's life, a period where he was completely blind, and where he was relying on his young collaborators for writing his papers).

Chebyshev, like Euler, was totally involved in investigations around the distribution of prime numbers. This is probably the most difficult problem in number theory. Euler wrote the following about this question, in his paper titled *Discovery of a very extraordinary law of numbers in relation to the sum of their divisors* [62], written in 1747:

> Mathematicians tried in vain, until now, to discover a certain order in the sequence of prime numbers, and we have reasons to think that this is a mystery which human mind will never be able to penetrate. To be convinced, it suffices to have a look at the tables of prime numbers, that a few persons have taken

the trouble to continue beyond one hundred thousand: one will primarily notice that there is no order and no rule there.

Chebyshev introduced completely new ideas in this area, developing methods that stand at the foundation of the field of analytic number theory. In 1848, he presented at the Saint Petersburg Academy of Sciences a paper titled *Sur la fonction qui détermine la totalité des nombres premiers inférieurs à une limite donnée* (On the function which determines the totality of prime numbers which are smaller than a given limit) [2, vol. I, p. 29–48] in which he starts the investigation of the function $\varphi(x)$ which to each positive integer x assigns the number of primes that are $\leq x$. He obtained the first important result on this function, correcting assertions that were made by Legendre before him, based on empirical observations and making them more precise. One of Chebyshev's results asserts that we can approximate this function by the integral $\int_2^x \frac{dx}{\log x}$, a result that was stated without proof by Gauss. The next year, Chebyshev defended a doctoral thesis on number theory titled *The theory of congruences*, which was published as a book which became a classic: it remained for several decades the only available Russian textbook on number theory. The next year (1850), Chebyshev published another paper on the distribution of primes, titled *Mémoire sur les nombres premieres* (Memoir on prime numbers) [2, Vol. I, p. 51–70], in which he gave a proof of the Bertrand *postulatum* which asserts that for any $n > 3$, there always exists a prime number between a and $2a - 2$.

Let us recall the important result of Euler, relating a quantity indexed by all natural integers, and another quantity indexed by the set of primes, which, in fact, is Euler's famous equality between the zeta function and an infinite product:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p=2}^{\infty} \left[1 - \left(\frac{1}{p^s} \right)^{-1} . \right]$$

Here, n describes the set of positive integers and p the set of primes. In Chebyshev's 1848 memoir which we just mentioned, he recalls that Euler proved that the series

$$\frac{1}{2^a} + \frac{1}{3^a} + \frac{1}{5^a} + \frac{1}{7^a} + \frac{1}{11^a} \dots ,$$

indexed by the primes, and the series

$$\frac{1}{2^a} + \frac{1}{3^a} + \frac{1}{4^a} + \frac{1}{5^a} + \frac{1}{6^a} \dots ,$$

indexed by all integers ≥ 2, converge or diverge for exactly the same values of a. (In fact, they both converge if and only if $a > 1$.) He then shows that this is not true in general, and he gives, as a counter-example, the series

$$\frac{1}{2 \log 2} + \frac{1}{3 \log 3} + \frac{1}{5 \log 5} + \frac{1}{7 \log 7} + \dots$$

indexed by the primes, which converges to a limit whose value is < 1.73, whereas the series

$$\frac{1}{2 \log 2} + \frac{1}{3 \log 3} + \frac{1}{4 \log 4} + \frac{1}{5 \log 5} + \frac{1}{6 \log 6} + \dots ,$$

indexed by all integers ≥ 2, diverges. He then addresses the general problem of finding a *criterion* for the convergence of a series indexed by prime numbers, and in the case where this series converges, to find a good approximation of its sum. He describes a general approach to this problem, which involves considering the function $\theta(x)$ which, for any number x, gives the sum of the logarithms of all primes that are at most x.

The examples we gave should be sufficient to show that Chebyshev's work on number theory, like all his other works, were very closely connected with his passion for approximation theory.

Conclusion

In many ways, Chebyshev's work is in the continuity of that of Euler. These two exceptional mathematicians were interested in similar problems, sometimes the same questions, always combining theory and practice. Chebyshev's influence in Saint Petersburg was still felt long after him, where most of the chairs of mathematics were occupied by his students or by mathematicians directly influenced by him. His impact was perceived far beyond the city of Saint Petersburg: some of his students and descendants founded mathematical schools in Kiev, Kharkov, Odessa, and in other centers of the Russian Empire. The combined efforts of Euler and Chebyshev are at the origin of the fame of the Saint Petersburg mathematical tradition. Their influence, beyond the city of Saint Petersburg and beyond Russia, is strongly felt worldwide.

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